CHAPTER

1

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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Introduction

Contents

What is Mechanics?

- Mechanics is the science which describes and predicts the conditions of rest or motion of bodies under the action of forces.

- Categories of Mechanics:
  - Rigid bodies
    - *Statics*
    - Dynamics
  - Deformable bodies
  - Fluids

- Mechanics is an applied science - it is not an abstract or pure science but does not have the empiricism found in other engineering sciences.

- Mechanics is the foundation of most engineering sciences and is an indispensable prerequisite to their study.

Fundamental Concepts

- *Space* - associated with the notion of the position of a point P given in terms of three coordinates measured from a reference point or origin.

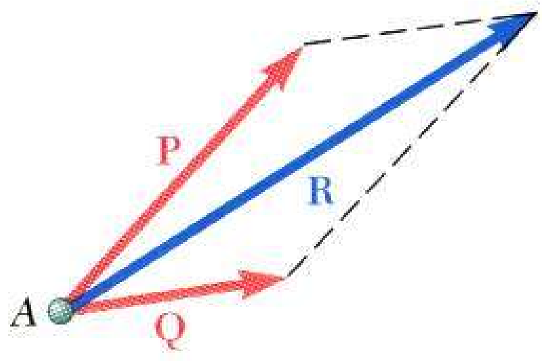
- *Time* - definition of an event requires specification of the time and position at which it occurred.

- *Mass* - used to characterize and compare bodies, e.g., response to earth's gravitational attraction and resistance to changes in translational motion.

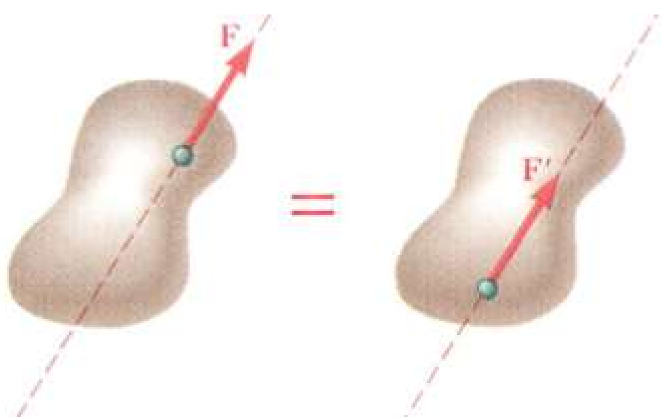
- *Force* - represents the action of one body on another. A force is characterized by its point of application, magnitude, and direction, i.e., a force is a vector quantity.

In Newtonian Mechanics, space, time, and mass are absolute concepts, independent of each other. Force, however, is not independent of the other three. The force acting on a body is related to the mass of the body and the variation of its velocity with time.

Fundamental Principles



- *Parallelogram Law*



- *Principle of Transmissibility*

- *Newton's First Law*: If the resultant force on a particle is zero, the particle will remain at rest or continue to move in a straight line.

- *Newton's Second Law*: A particle will have an acceleration proportional to a nonzero resultant applied force.

$$\vec{F} = m\vec{a}$$

- *Newton's Third Law*: The forces of action and reaction between two particles have the same magnitude and line of action with opposite sense.

- *Newton's Law of Gravitation*: Two particles are attracted with equal and opposite forces,

$$F = G\frac{Mm}{r^2} \qquad W = mg, \quad g = \frac{GM}{R^2}$$

Systems of Units

- *Kinetic Units*: length, time, mass, and force.

- Three of the kinetic units, referred to as *basic units*, may be defined arbitrarily. The fourth unit, referred to as a *derived unit*, must have a definition compatible with Newton's 2nd Law,

$$\vec{F} = m\vec{a}$$

- *International System of Units* (SI):
The basic units are length, time, and mass which are arbitrarily defined as the meter (m), second (s), and kilogram (kg). Force is the derived unit,

$$F = ma$$

$$1\,\mathrm{N} = (1\,\mathrm{kg})\left(1\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)$$

- *U.S. Customary Units*:
The basic units are length, time, and force which are arbitrarily defined as the foot (ft), second (s), and pound (lb). Mass is the derived unit,

$$m = \frac{F}{a}$$

$$1\,\mathrm{slug} = \frac{1\,\mathrm{lb}}{1\,\mathrm{ft/s}^2}$$

Method of Problem Solution

- *Problem Statement*:
  Includes given data, specification of what is to be determined, and a figure showing all quantities involved.

- *Free-Body Diagrams*:
  Create separate diagrams for each of the bodies involved with a clear indication of all forces acting on each body.

- *Fundamental Principles*:
  The six fundamental principles are applied to express the conditions of rest or motion of each body. The rules of algebra are applied to solve the equations for the unknown quantities.

- *Solution Check*:
  - Test for errors in reasoning by verifying that the units of the computed results are correct,
  - test for errors in computation by substituting given data and computed results into previously unused equations based on the six principles,
  - **always** apply experience and physical intuition to assess whether results seem "reasonable"

Numerical Accuracy

- The accuracy of a solution depends on 1) accuracy of the given data, and 2) accuracy of the computations performed. The solution cannot be more accurate than the less accurate of these two.
- The use of hand calculators and computers generally makes the accuracy of the computations much greater than the accuracy of the data. Hence, the solution accuracy is usually limited by the data accuracy.
- As a general rule for engineering problems, the data are seldom known with an accuracy greater than 0.2%. Therefore, it is usually appropriate to record parameters beginning with “1” with four digits and with three digits in all other cases, i.e., 40.2 lb and 15.58 lb.

CHAPTER

2

VECTOR MECHANICS FOR ENGINEERS:
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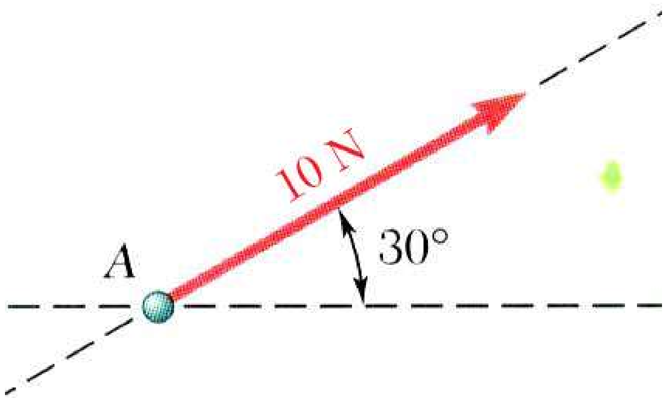
Statics of Particles

Contents

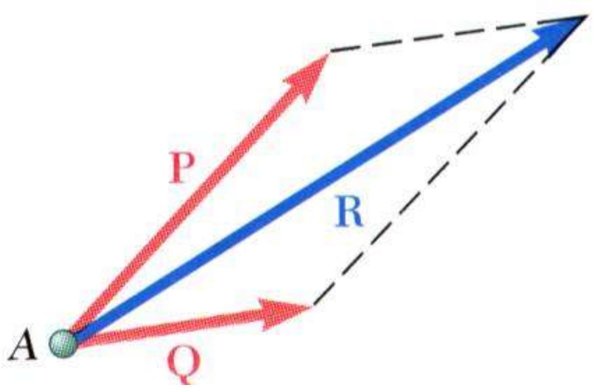
Introduction

- The objective for the current chapter is to investigate the effects of forces on particles:
  - replacing multiple forces acting on a particle with a single equivalent or *resultant* force,
  - relations between forces acting on a particle that is in a state of *equilibrium*.
- The focus on *particles* does not imply a restriction to miniscule bodies. Rather, the study is restricted to analyses in which the size and shape of the bodies is not significant so that all forces may be assumed to be applied at a single point.

Resultant of Two Forces



- force: action of one body on another; characterized by its *point of application*, *magnitude*, *line of action*, and *sense*.

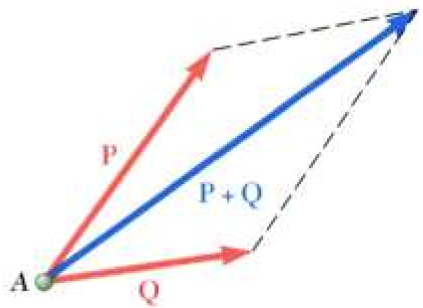


- Experimental evidence shows that the combined effect of two forces may be represented by a single *resultant* force.

- The resultant is equivalent to the diagonal of a parallelogram which contains the two forces in adjacent legs.

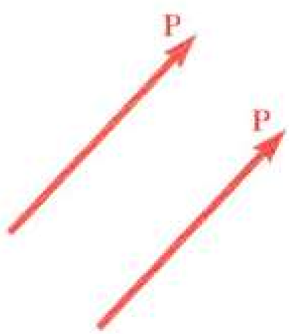
- Force is a *vector* quantity.

Vectors

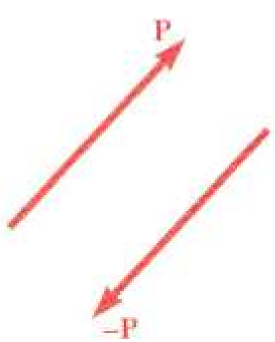


- *Vector*: parameters possessing magnitude and direction which add according to the parallelogram law. Examples: displacements, velocities, accelerations.

- *Scalar*: parameters possessing magnitude but not direction. Examples: mass, volume, temperature



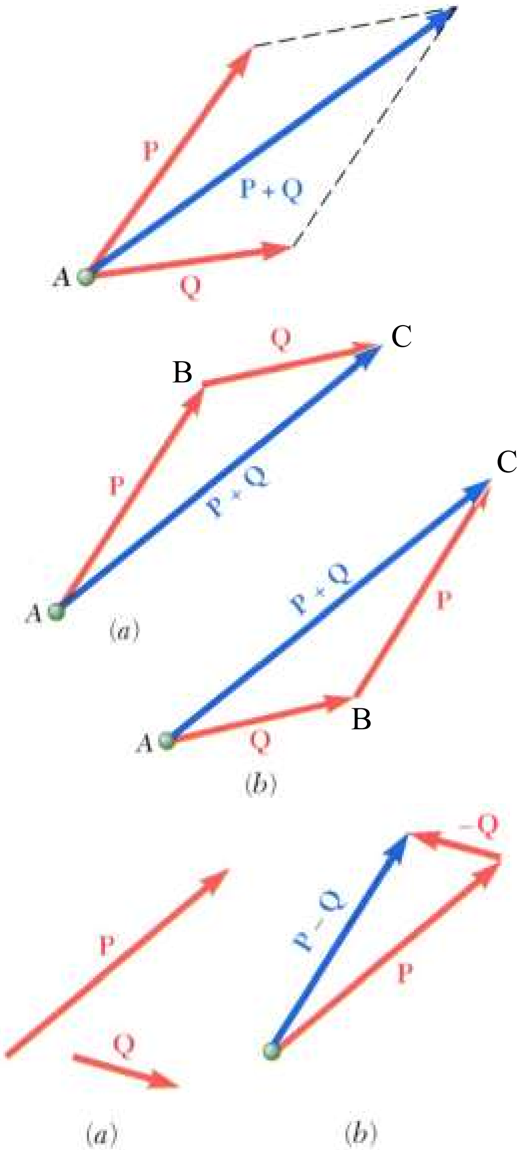
- Vector classifications:
  - *Fixed* or *bound* vectors have well defined points of application that cannot be changed without affecting an analysis.
  - *Free* vectors may be freely moved in space without changing their effect on an analysis.
  - *Sliding* vectors may be applied anywhere along their line of action without affecting an analysis.



- *Equal* vectors have the same magnitude and direction.

- *Negative* vector of a given vector has the same magnitude and the opposite direction.

Addition of Vectors



- Trapezoid rule for vector addition
- Triangle rule for vector addition
- Law of cosines,

$$R^2 = P^2 + Q^2 - 2PQ\cos B$$
$$\vec{R} = \vec{P} + \vec{Q}$$

- Law of sines,

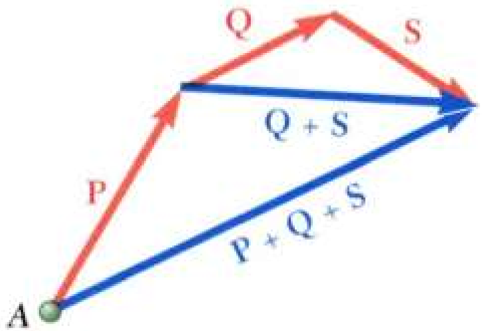
$$\frac{\sin A}{Q} = \frac{\sin B}{R} = \frac{\sin C}{P}$$

- Vector addition is commutative,

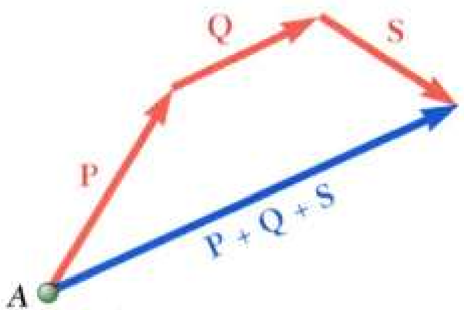
$$\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$$

- Vector subtraction

Addition of Vectors

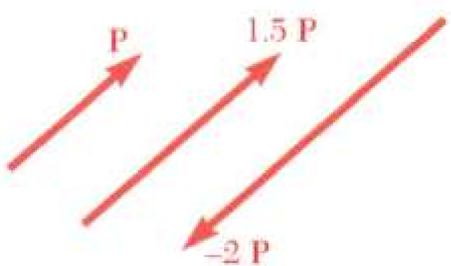


- Addition of three or more vectors through repeated application of the triangle rule



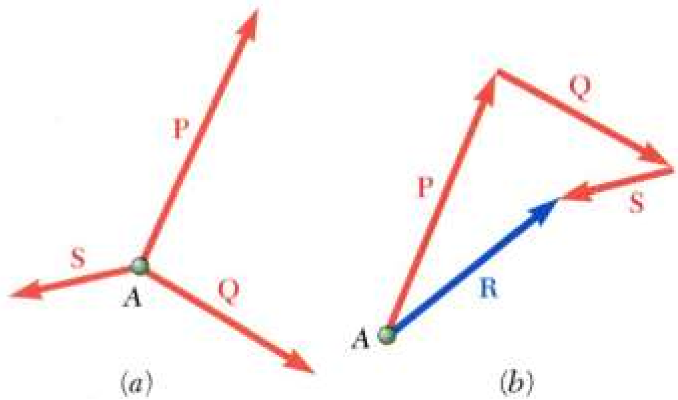
- The polygon rule for the addition of three or more vectors.
- Vector addition is associative,

$$\vec{P}+\vec{Q}+\vec{S}=\left(\vec{P}+\vec{Q}\right)+\vec{S}=\vec{P}+\left(\vec{Q}+\vec{S}\right)$$



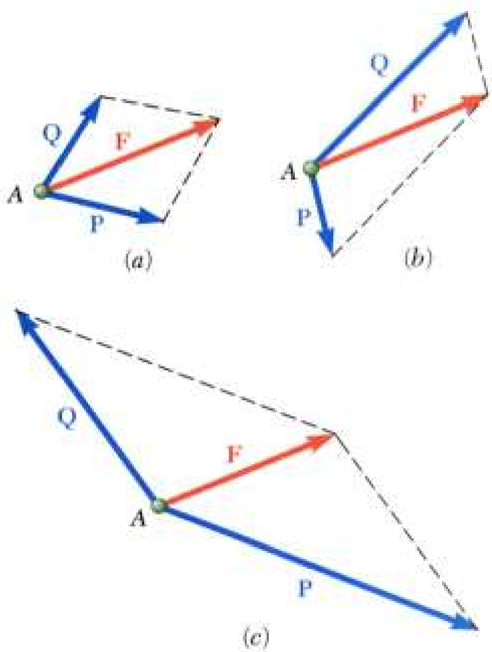
- Multiplication of a vector by a scalar

Resultant of Several Concurrent Forces



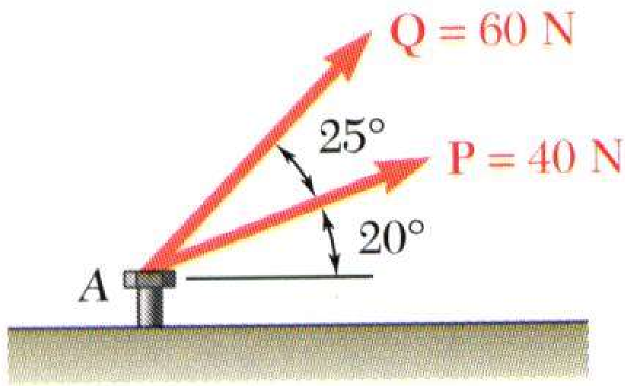
- *Concurrent forces*: set of forces which all pass through the same point.

  A set of concurrent forces applied to a particle may be replaced by a single resultant force which is the vector sum of the applied forces.



- *Vector force components*: two or more force vectors which, together, have the same effect as a single force vector.

Sample Problem 2.1

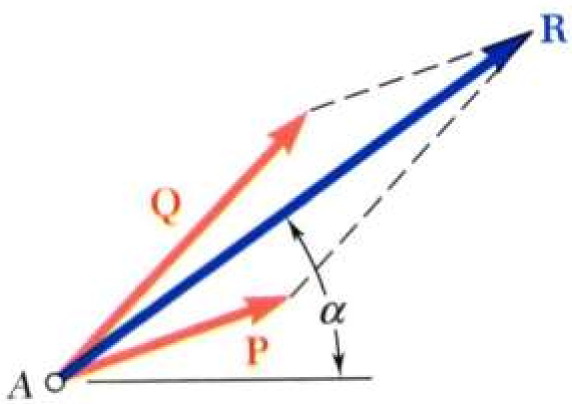


The two forces act on a bolt at *A*. Determine their resultant.

SOLUTION:

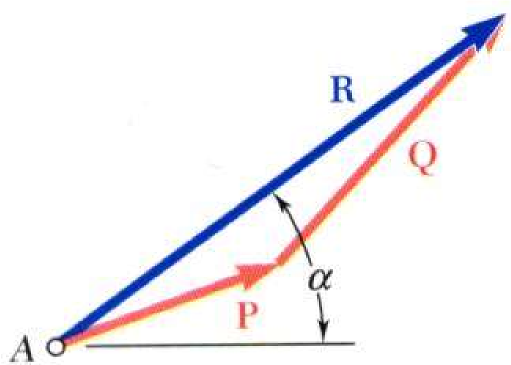
- Graphical solution - construct a parallelogram with sides in the same direction as **P** and **Q** and lengths in proportion. Graphically evaluate the resultant which is equivalent in direction and proportional in magnitude to the the diagonal.
- Trigonometric solution - use the triangle rule for vector addition in conjunction with the law of cosines and law of sines to find the resultant.

Sample Problem 2.1



- Graphical solution - A parallelogram with sides equal to **P** and **Q** is drawn to scale. The magnitude and direction of the resultant or of the diagonal to the parallelogram are measured,

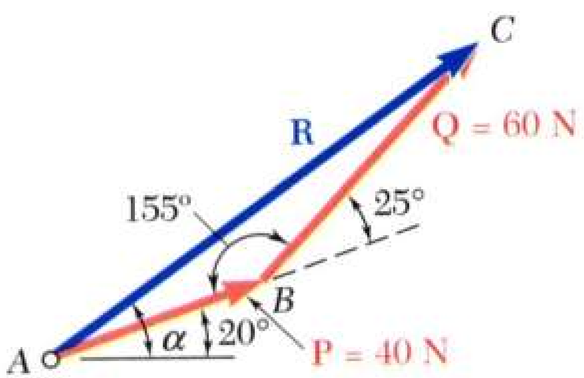
$$\boxed{\mathbf{R} = 98\,\mathrm{N} \quad \alpha = 35^\circ}$$



- Graphical solution - A triangle is drawn with **P** and **Q** head-to-tail and to scale. The magnitude and direction of the resultant or of the third side of the triangle are measured,

$$\boxed{\mathbf{R} = 98\,\mathrm{N} \quad \alpha = 35^\circ}$$

Sample Problem 2.1



- Trigonometric solution - Apply the triangle rule.

From the Law of Cosines,

$$R^2 = P^2 + Q^2 - 2PQ\cos B$$
$$= (40\text{N})^2 + (60\text{N})^2 - 2(40\text{N})(60\text{N})\cos 155^\circ$$

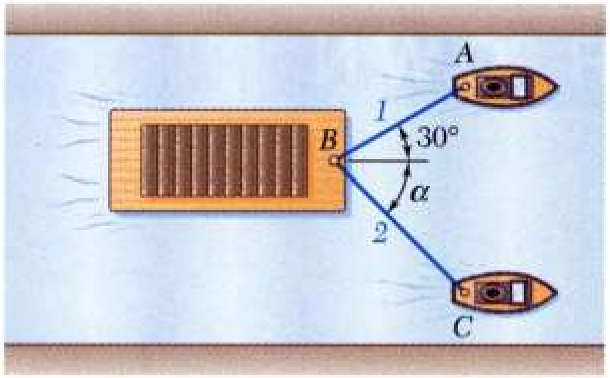
$$\boxed{R = 97.73\text{N}}$$

From the Law of Sines,

$$\frac{\sin A}{Q} = \frac{\sin B}{R}$$

$$\sin A = \sin B \frac{Q}{R}$$
$$= \sin 155^\circ \frac{60\text{N}}{97.73\text{N}}$$
$$A = 15.04^\circ$$
$$\alpha = 20^\circ + A$$
$$\boxed{\alpha = 35.04^\circ}$$

Sample Problem 2.2



A barge is pulled by two tugboats. If the resultant of the forces exerted by the tugboats is 5000 N directed along the axis of the barge, determine

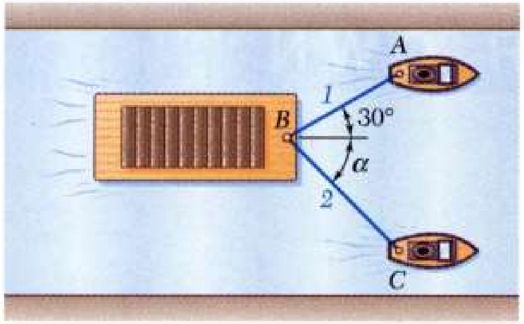
a) the tension in each of the ropes for $\alpha = 45^{o}$,

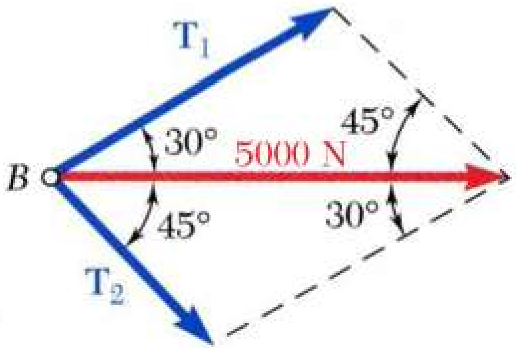
b) the value of α for which the tension in rope 2 is a minimum.

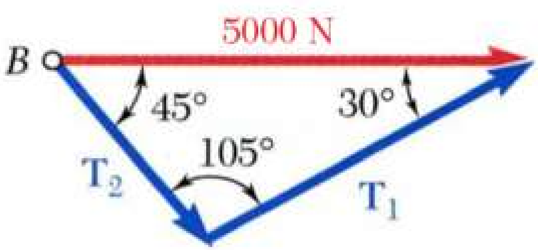
SOLUTION:

- Find a graphical solution by applying the Parallelogram Rule for vector addition. The parallelogram has sides in the directions of the two ropes and a diagonal in the direction of the barge axis and length proportional to 5000 N.
- Find a trigonometric solution by applying the Triangle Rule for vector addition. With the magnitude and direction of the resultant known and the directions of the other two sides parallel to the ropes given, apply the Law of Sines to find the rope tensions.
- The angle for minimum tension in rope 2 is determined by applying the Triangle Rule and observing the effect of variations in α.

Sample Problem 2.2







- Graphical solution - Parallelogram Rule with known resultant direction and magnitude, known directions for sides.

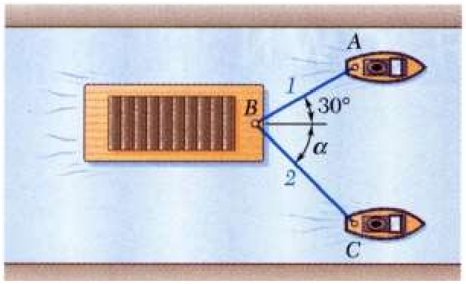
$$\boxed{T_1 = 3700\,\text{N} \quad T_2 = 2600\,\text{N}}$$

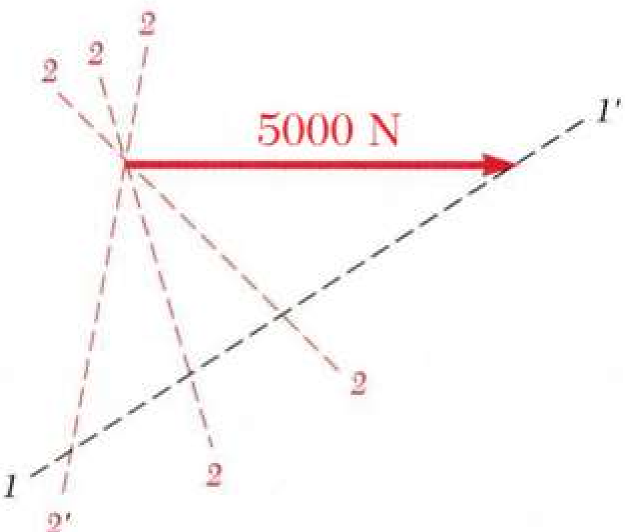
- Trigonometric solution - Triangle Rule with Law of Sines

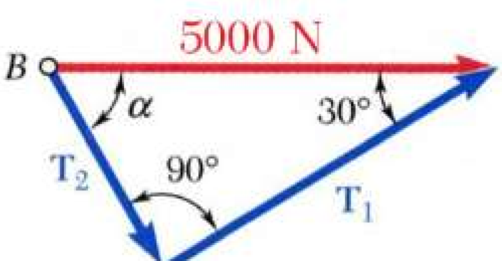
$$\frac{T_1}{\sin 45°} = \frac{T_2}{\sin 30°} = \frac{5000\,\text{N}}{\sin 105°}$$

$$\boxed{T_1 = 3660\,\text{N} \quad T_2 = 2590\,\text{N}}$$

Sample Problem 2.2







- The angle for minimum tension in rope 2 is determined by applying the Triangle Rule and observing the effect of variations in α.

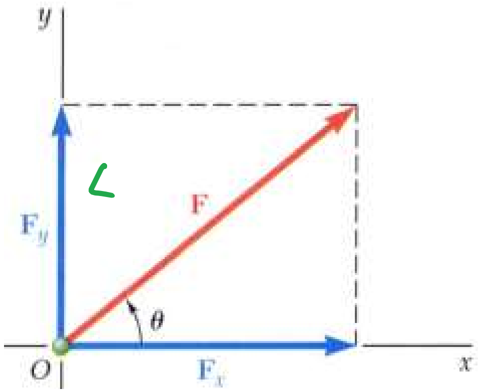
- The minimum tension in rope 2 occurs when $\mathbf{T_1}$ and $\mathbf{T_2}$ are perpendicular.

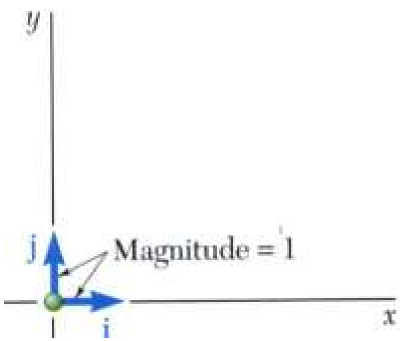
$$T_2 = (5000\,\text{N})\sin 30° \qquad \boxed{T_2 = 2500\,\text{N}}$$

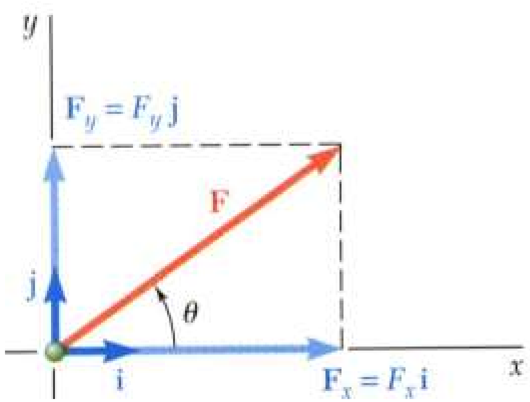
$$T_1 = (5000\,\text{N})\cos 30° \qquad \boxed{T_1 = 4330\,\text{N}}$$

$$\alpha = 90° - 30° \qquad \boxed{\alpha = 60°}$$

Rectangular Components of a Force: Unit Vectors







- May resolve a force vector into perpendicular components so that the resulting parallelogram is a rectangle. $\vec{F}_x$ and $\vec{F}_y$ are referred to as *rectangular vector components* and

$$\vec{F} = \vec{F}_x + \vec{F}_y$$

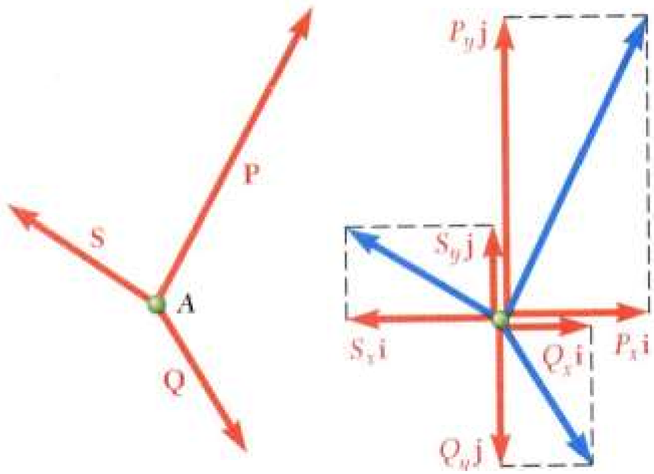
- Define perpendicular *unit vectors* $\vec{i}$ and $\vec{j}$ which are parallel to the *x* and *y* axes.

- Vector components may be expressed as products of the unit vectors with the scalar magnitudes of the vector components.

$$\vec{F} = F_x\vec{i} + F_y\vec{j}$$

F_x and F_y are referred to as the *scalar components* of $\vec{F}$

Addition of Forces by Summing Components

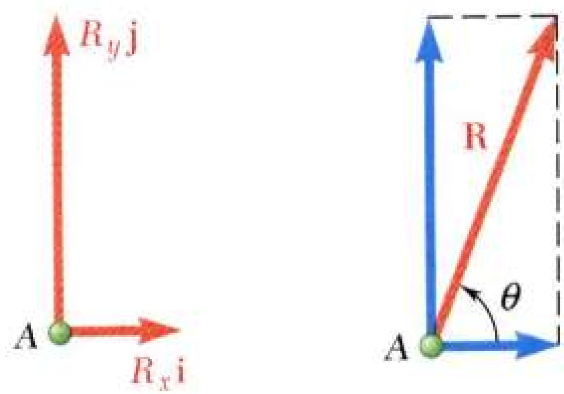


- Wish to find the resultant of 3 or more concurrent forces,

$$\vec{R} = \vec{P} + \vec{Q} + \vec{S}$$

- Resolve each force into rectangular components

$$\begin{aligned} R_x\vec{i} + R_y\vec{j} &= P_x\vec{i} + P_y\vec{j} + Q_x\vec{i} + Q_y\vec{j} + S_x\vec{i} + S_y\vec{j} \\ &= \left(P_x + Q_x + S_x\right)\vec{i} + \left(P_y + Q_y + S_y\right)\vec{j} \end{aligned}$$



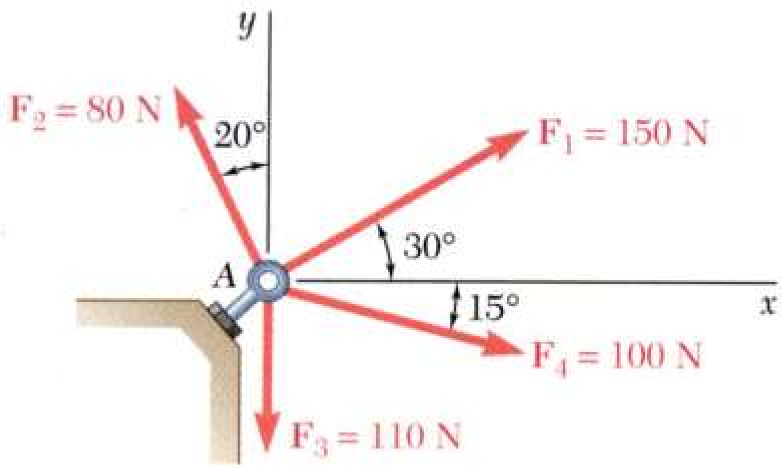
- The scalar components of the resultant are equal to the sum of the corresponding scalar components of the given forces.

$$\begin{aligned} R_x &= P_x + Q_x + S_x \\ &= \sum F_x \end{aligned} \qquad \begin{aligned} R_y &= P_y + Q_y + S_y \\ &= \sum F_y \end{aligned}$$

- To find the resultant magnitude and direction,

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$

Sample Problem 2.3

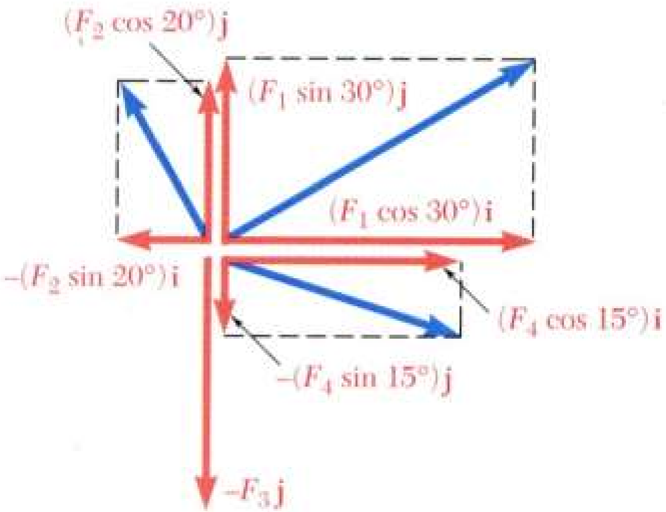


Four forces act on bolt *A* as shown. Determine the resultant of the force on the bolt.

SOLUTION:

- Resolve each force into rectangular components.
- Determine the components of the resultant by adding the corresponding force components.
- Calculate the magnitude and direction of the resultant.

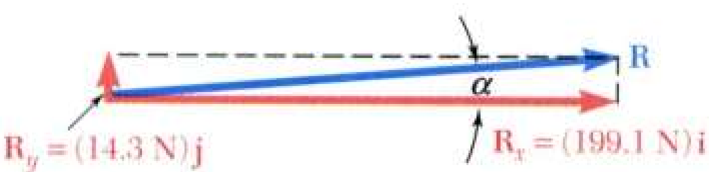
Sample Problem 2.3



SOLUTION:

- Resolve each force into rectangular components.

force	*mag*	*x – comp*	*y – comp*
$\vec{F}_1$	150	+129.9	+75.0
$\vec{F}_2$	80	−27.4	+75.2
$\vec{F}_3$	110	0	−110.0
$\vec{F}_4$	100	+96.6	−25.9
		$R_x = +199.1$	$R_y = +14.3$



- Determine the components of the resultant by adding the corresponding force components.

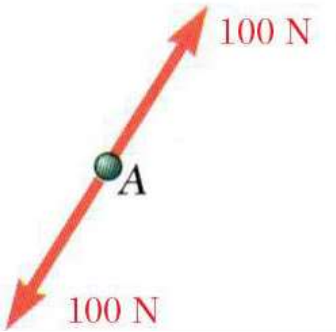
- Calculate the magnitude and direction.

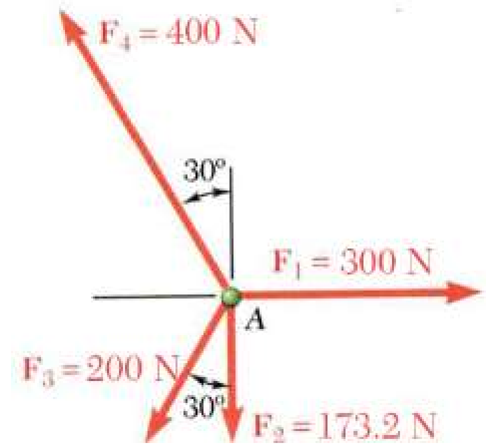
$$R = \sqrt{199.1^2 + 14.3^2} \qquad \boxed{R = 199.6\text{N}}$$

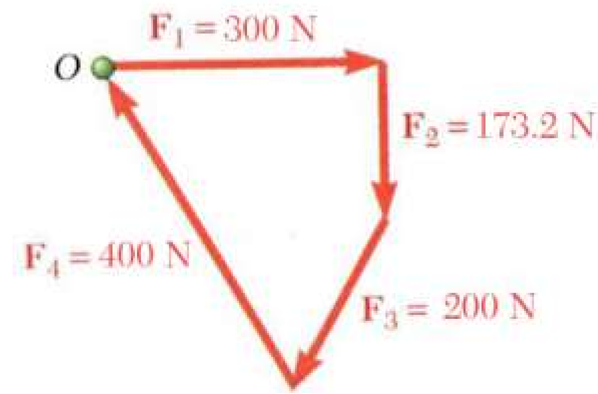
$$\tan\alpha = \frac{14.3\,\text{N}}{199.1\,\text{N}} \qquad \boxed{\alpha = 4.1°}$$

Equilibrium of a Particle

- When the resultant of all forces acting on a particle is zero, the particle is in *equilibrium*.
- *Newton's First Law*: If the resultant force on a particle is zero, the particle will remain at rest or will continue at constant speed in a straight line.





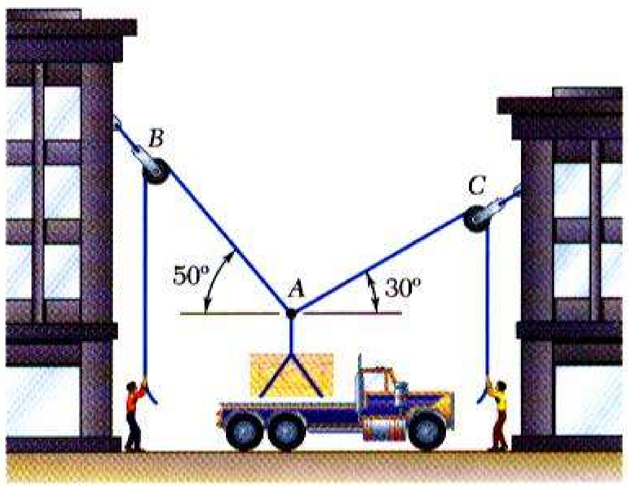


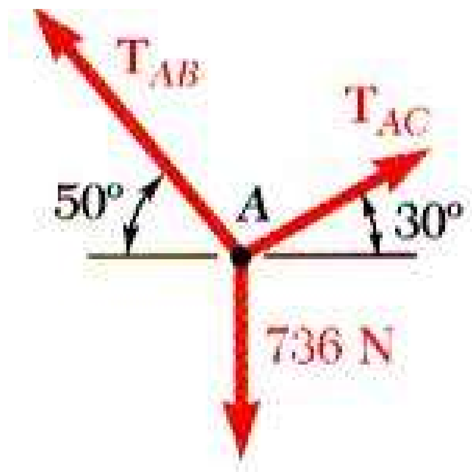
- Particle acted upon by two forces:
  - equal magnitude
  - same line of action
  - opposite sense
- Particle acted upon by three or more forces:
  - graphical solution yields a closed polygon
  - algebraic solution

$$\vec{R} = \sum \vec{F} = 0$$

$$\sum F_x = 0 \qquad \sum F_y = 0$$

Free-Body Diagrams

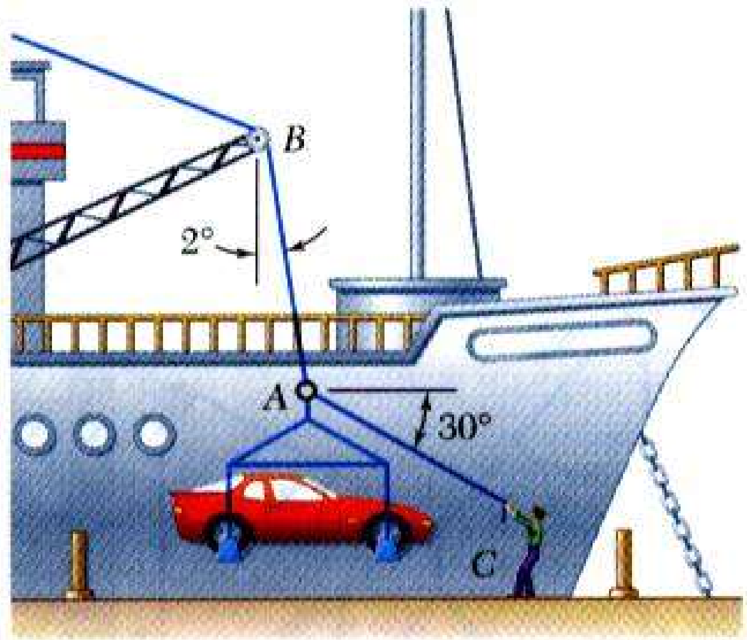




Space Diagram: A sketch showing the physical conditions of the problem.

Free-Body Diagram: A sketch showing only the forces on the selected particle.

Sample Problem 2.4

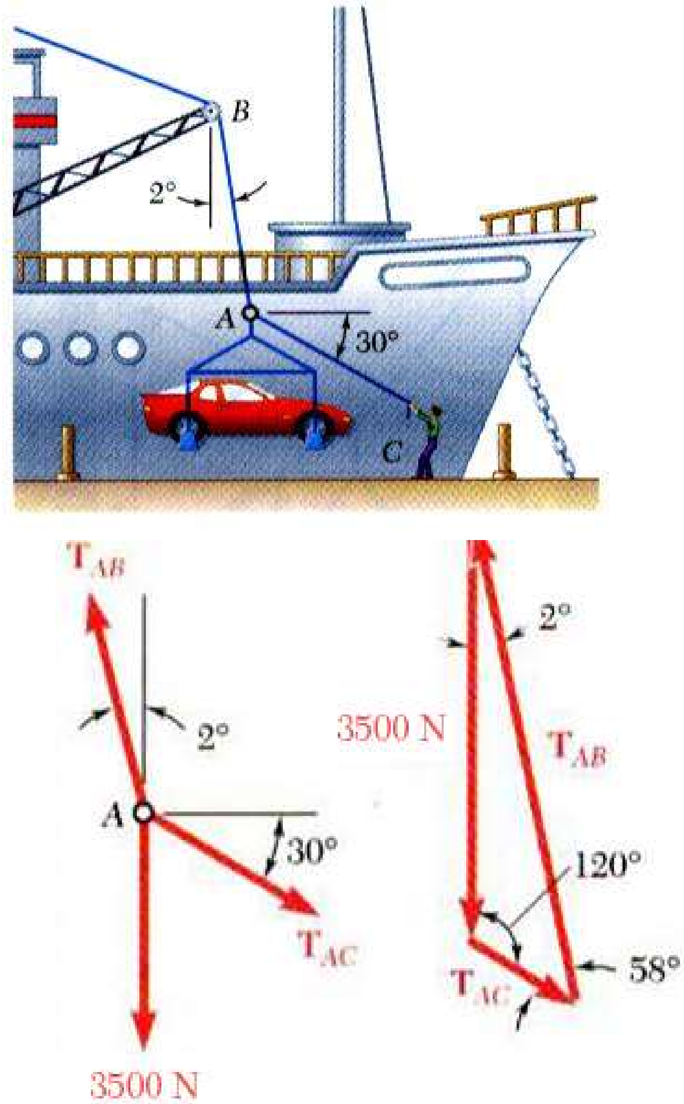


In a ship-unloading operation, a 3500-N automobile is supported by a cable. A rope is tied to the cable and pulled to center the automobile over its intended position. What is the tension in the rope?

SOLUTION:

- Construct a free-body diagram for the particle at the junction of the rope and cable.
- Apply the conditions for equilibrium by creating a closed polygon from the forces applied to the particle.
- Apply trigonometric relations to determine the unknown force magnitudes.

Sample Problem 2.4



SOLUTION:

- Construct a free-body diagram for the particle at *A*.
- Apply the conditions for equilibrium.
- Solve for the unknown force magnitudes.

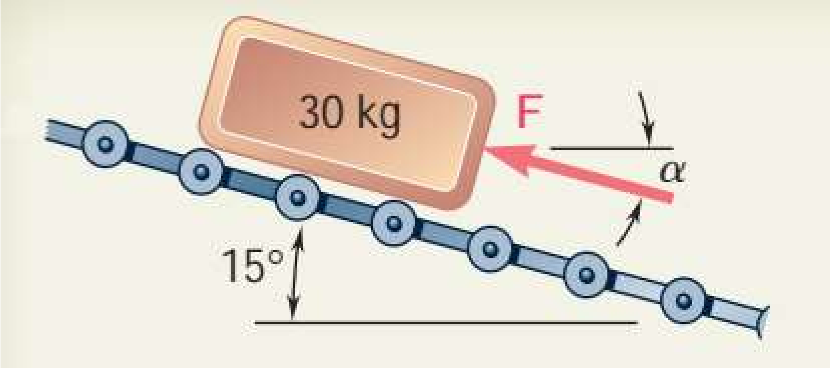
$$\frac{T_{AB}}{\sin 120°} = \frac{T_{AC}}{\sin 2°} = \frac{3500\,\text{N}}{\sin 58°}$$

$$T_{AB} = 3570\,\text{N}$$

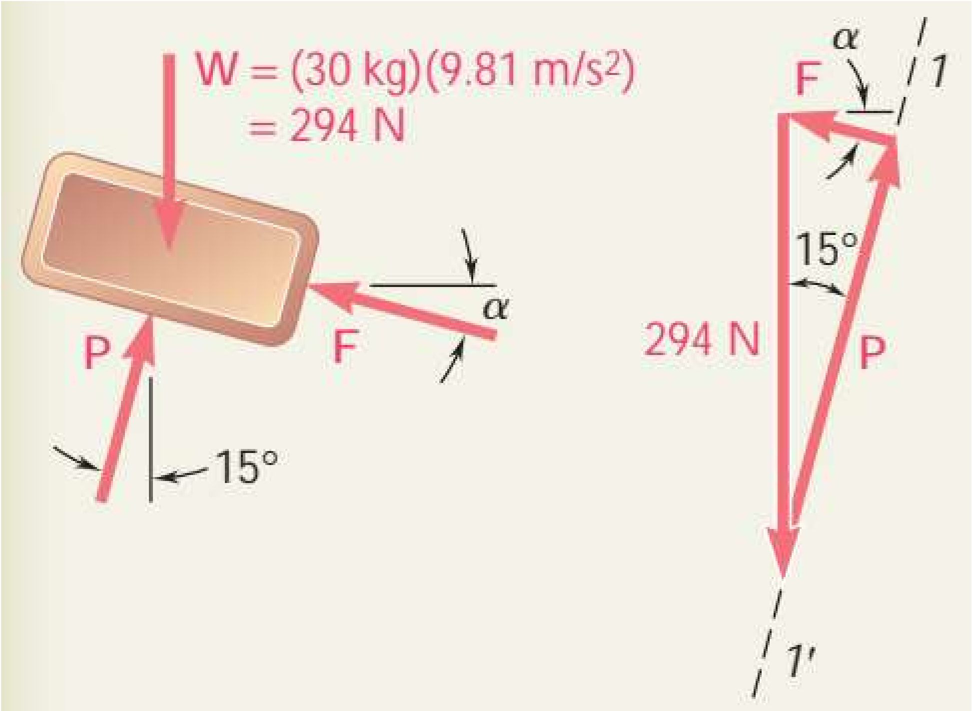
$$\boxed{T_{AC} = 144\,\text{N}}$$

Sample Problem 2.5

Determine the magnitude and direction of the smallest force **F** which will maintain the package shown in equilibrium. Note that the force exerted by the rollers on the package is perpendicular to the incline.



Sample Problem 2.5



Sample Problem 2.5

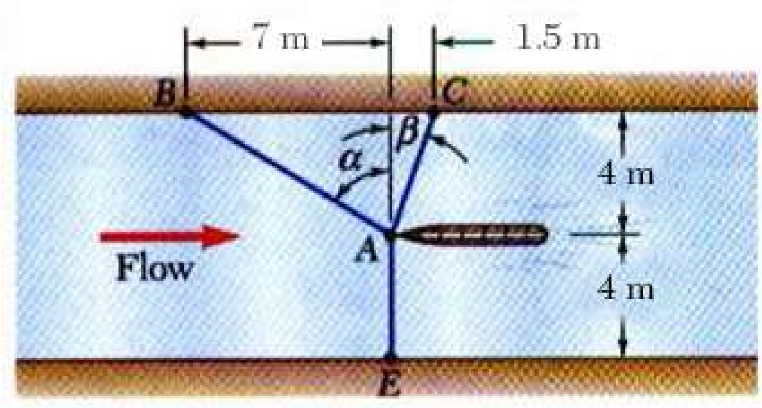
Free-Body Diagram. We choose the package as a free body, assuming that it can be treated as a particle. We draw the corresponding free-body diagram.

Equilibrium Condition. Since only three forces act on the free body, we draw a force triangle to express that it is in equilibrium. Line *1-1′* represents the known direction of **P**. In order to obtain the minimum value of the force **F**, we choose the direction of **F** perpendicular to that of **P**. From the geometry of the triangle obtained, we find

$$F = (294\ \text{N})\sin 15° = 76.1\ \text{N} \qquad a = 15°$$

$$\mathbf{F} = 76.1\ \text{N} \ b\ 15°$$

Sample Problem 2.6



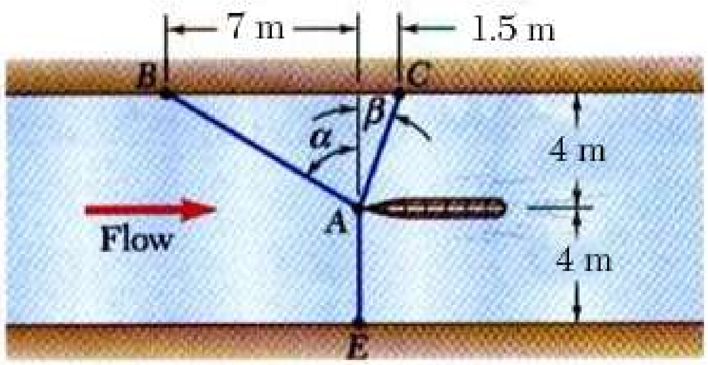
It is desired to determine the drag force at a given speed on a prototype sailboat hull. A model is placed in a test channel and three cables are used to align its bow on the channel centerline. For a given speed, the tension is 40 N in cable *AB* and 60 N in cable *AE*.

Determine the drag force exerted on the hull and the tension in cable *AC*.

SOLUTION:

- Choosing the hull as the free body, draw a free-body diagram.
- Express the condition for equilibrium for the hull by writing that the sum of all forces must be zero.
- Resolve the vector equilibrium equation into two component equations. Solve for the two unknown cable tensions.

Sample Problem 2.6

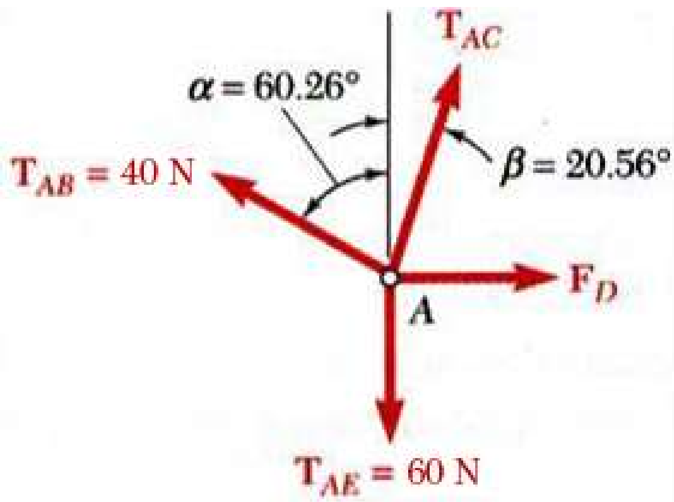


SOLUTION:

- Choosing the hull as the free body, draw a free-body diagram.

$$\tan\alpha = \frac{7\text{ m}}{4\text{ m}} = 1.75 \qquad \tan\beta = \frac{1.5\text{ m}}{4\text{ m}} = 0.375$$

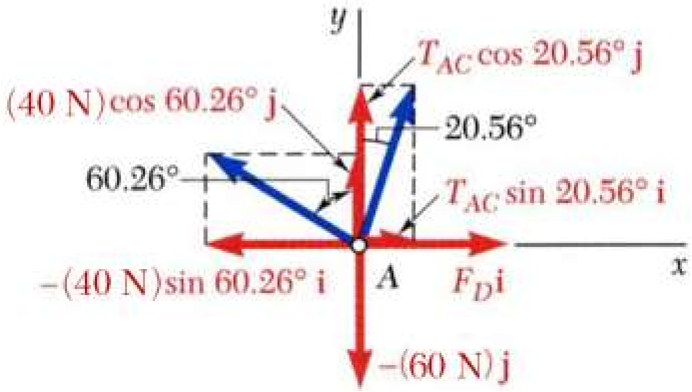
$$\alpha = 60.25° \qquad \beta = 20.56°$$

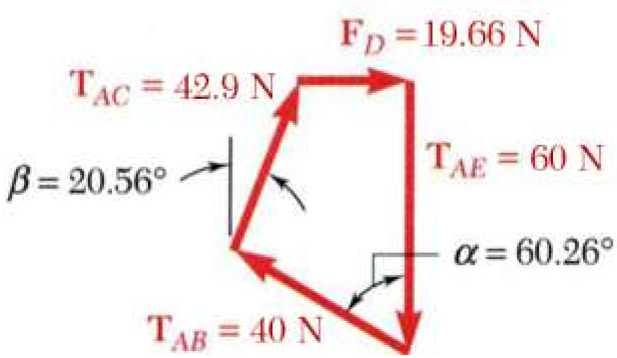


- Express the condition for equilibrium for the hull by writing that the sum of all forces must be zero.

$$\vec{R} = \vec{T}_{AB} + \vec{T}_{AC} + \vec{T}_{AE} + \vec{F}_D = 0$$

Sample Problem 2.6





- Resolve the vector equilibrium equation into two component equations. Solve for the two unknown cable tensions.

$$\vec{T}_{AB} = -(40\text{ N})\sin 60.26^\circ\,\vec{i} + (40\text{ N})\cos 60.26^\circ\,\vec{j}$$
$$= -(34.73\text{ N})\vec{i} + (19.84\text{ N})\vec{j}$$

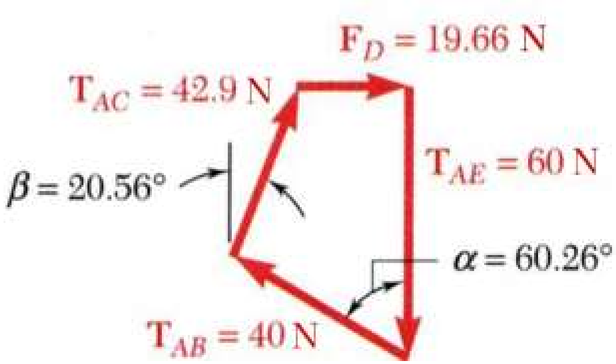
$$\vec{T}_{AC} = T_{AC}\sin 20.56^\circ\,\vec{i} + T_{AC}\cos 20.56^\circ\,\vec{j}$$
$$= 0.3512\,T_{AC}\,\vec{i} + 0.9363\,T_{AC}\,\vec{j}$$

$$\vec{T} = -(60\text{ N})\vec{j}$$

$$\vec{F}_D = F_D\,\vec{i}$$

$$\vec{R} = 0$$
$$= (-34.73 + 0.3512\,T_{AC} + F_D)\vec{i}$$
$$+ (19.84 + 0.9363\,T_{AC} - 60)\vec{j}$$

Sample Problem 2.6



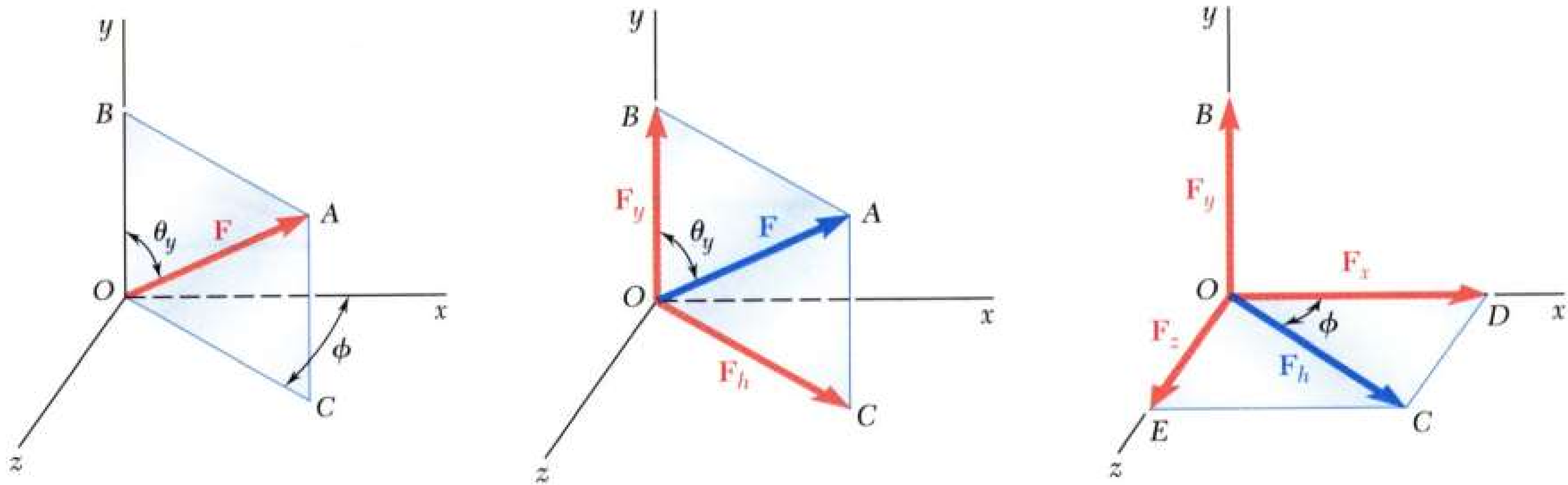
$$\vec{R} = 0$$
$$= \left(-34.73 + 0.3512 T_{AC} + F_D\right)\vec{i}$$
$$+ \left(19.84 + 0.9363 T_{AC} - 60\right)\vec{j}$$

This equation is satisfied only if each component of the resultant is equal to zero.

$$\left(\sum F_x = 0\right) \quad 0 = -34.73 + 0.3512 T_{AC} + F_D$$
$$\left(\sum F_y = 0\right) \quad 0 = 19.84 + 0.9363 T_{AC} - 60$$

$$\boxed{\begin{aligned} T_{AC} &= +42.9\text{ N} \\ F_D &= +19.66\text{ N} \end{aligned}}$$

Rectangular Components in Space



- The vector $\vec{F}$ is contained in the plane $OBAC$.

- Resolve $\vec{F}$ into horizontal and vertical components.

$$F_y = F\cos\theta_y$$

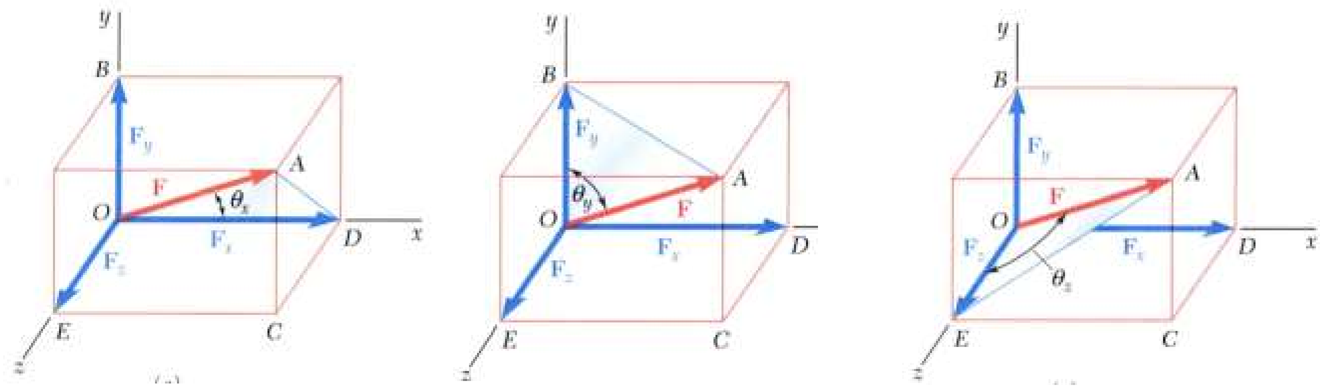
$$F_h = F\sin\theta_y$$

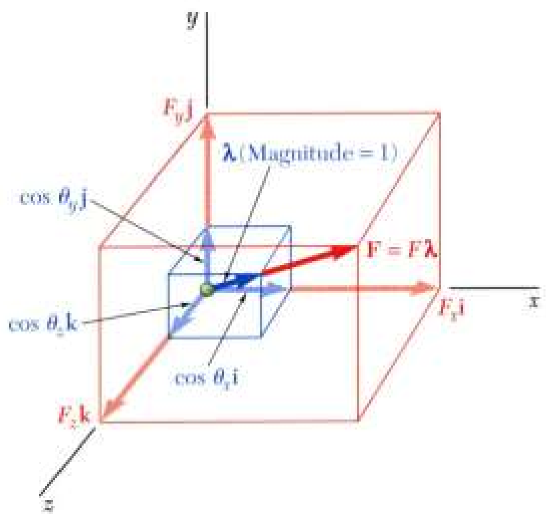
- Resolve F_h into rectangular components.

$$F_x = F_h\cos\phi$$
$$= F\sin\theta_y\cos\phi$$

$$F_y = F_h\sin\phi$$
$$= F\sin\theta_y\sin\phi$$

Rectangular Components in Space





- With the angles between $\vec{F}$ and the axes,

$$F_x = F\cos\theta_x \quad F_y = F\cos\theta_y \quad F_z = F\cos\theta_z$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$= F\left(\cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k}\right)$$

$$= F\vec{\lambda}$$

$$\vec{\lambda} = \cos\theta_x\vec{i} + \cos\theta_y\vec{j} + \cos\theta_z\vec{k}$$

- $\vec{\lambda}$ is a unit vector along the line of action of $\vec{F}$ and $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$ are the direction cosines for $\vec{F}$

Rectangular Components in Space

Magnitude of a vector using x, y, and z coordinates:

Show that

$$\boxed{|F| = \sqrt{F_x^{\,2} + F_y^{\,2} + F_z^{\,2}}} \qquad \text{(Equation 2.18)}$$

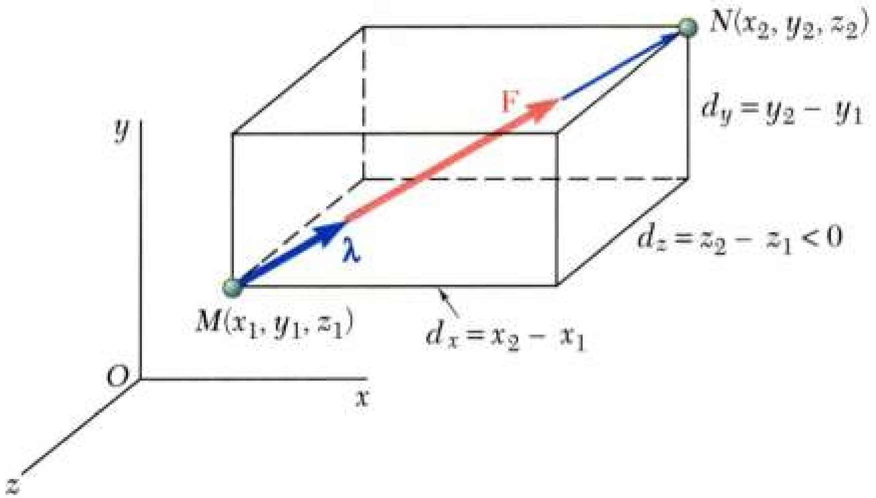
Also show that

$$\boxed{\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1} \qquad \text{(Equation 2.20)}$$

Rectangular Components in Space

Direction of the force is defined by the location of two points,

$$M(x_1, y_1, z_1) \text{ and } N(x_2, y_2, z_2)$$



$$\vec{d} = \text{vector joining } M \text{ and } N$$

$$= d_x \vec{i} + d_y \vec{j} + d_z \vec{k}$$

$$d_x = x_2 - x_1 \quad d_y = y_2 - y_1 \quad d_z = z_2 - z_1$$

$$\vec{F} = F \vec{\lambda}$$

$$\vec{\lambda} = \frac{1}{d}\left(d_x \vec{i} + d_y \vec{j} + d_z \vec{k}\right)$$

$$F_x = \frac{F d_x}{d} \quad F_y = \frac{F d_y}{d} \quad F_z = \frac{F d_z}{d}$$

Determining Resultants in Space

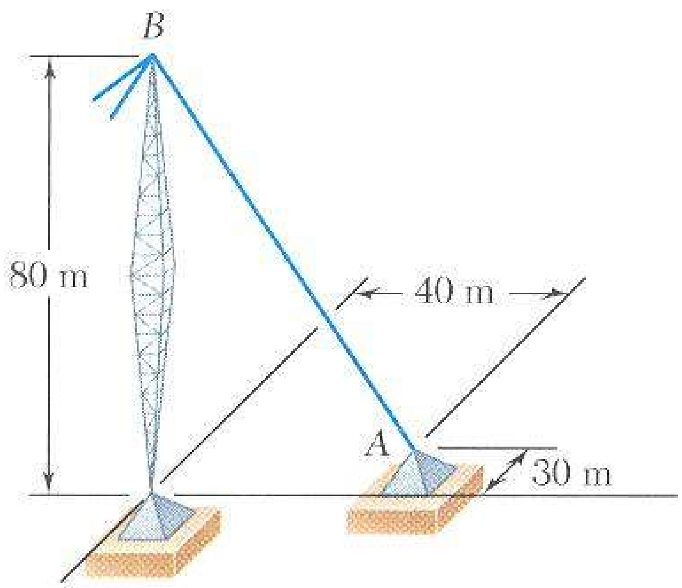
Determining resultants using x, y, and z rectangular components

Procedure:

1. Express each force using unit vectors
2. Add all x components for the total (resultant) x component, i.e., $R_x = \Sigma F_x$
3. Add all y components for the total (resultant) y component, i.e., $R_y = \Sigma F_y$
4. Add all z components for the total (resultant) z component, i.e., $R_z = \Sigma F_z$
5. Express the final result as:

$$\boxed{\overline{R} = R_x i + R_y j + R_z k}$$

Sample Problem 2.7



The tension in the guy wire is 2500 N. Determine:

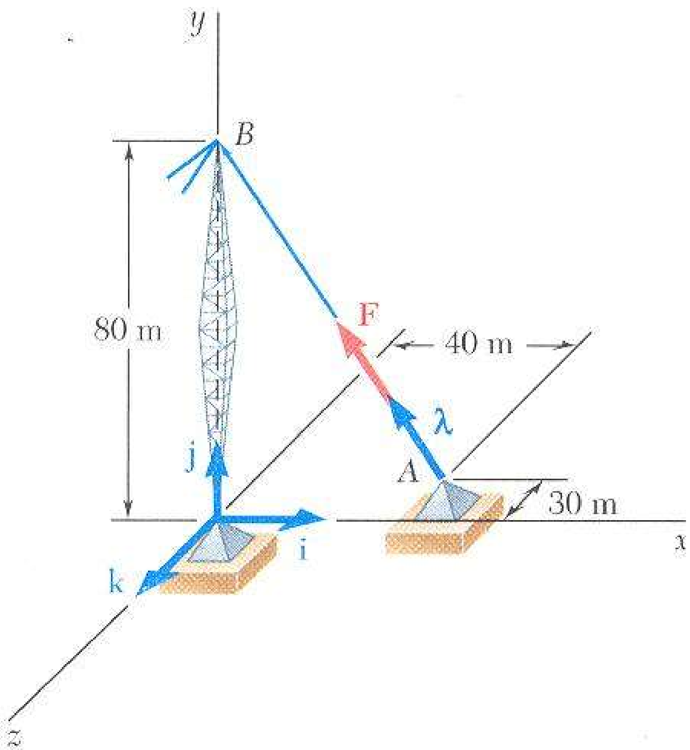
a) components F_x, F_y, F_z of the force acting on the bolt at A,

b) the angles θ_x, θ_y, θ_z defining the direction of the force

SOLUTION:

- Based on the relative locations of the points A and B, determine the unit vector pointing from A towards B.
- Apply the unit vector to determine the components of the force acting on A.
- Noting that the components of the unit vector are the direction cosines for the vector, calculate the corresponding angles.

Sample Problem 2.7



SOLUTION:

- Determine the unit vector pointing from A towards B.

$$\overrightarrow{AB} = (-40\,\text{m})\vec{i} + (80\,\text{m})\vec{j} + (30\,\text{m})\vec{k}$$

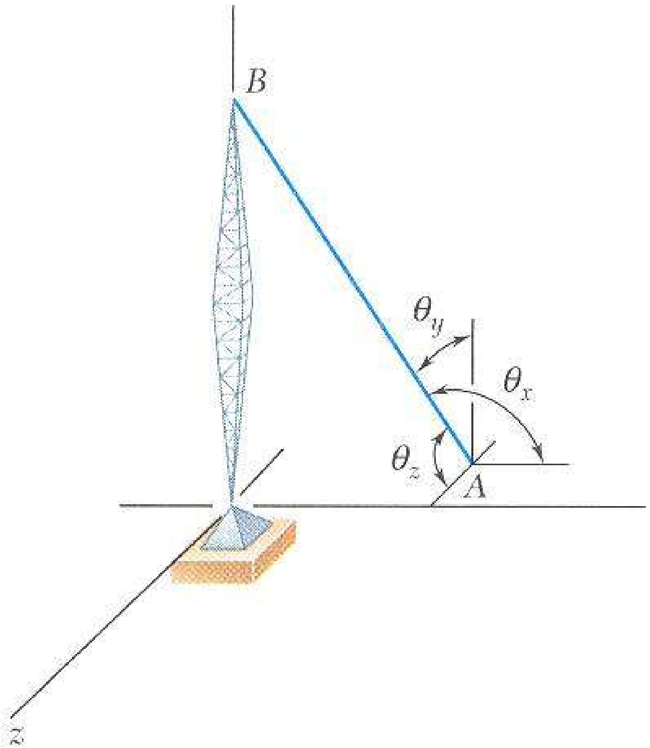
$$AB = \sqrt{(-40\,\text{m})^2 + (80\,\text{m})^2 + (30\,\text{m})^2} = 94.3\,\text{m}$$

$$\vec{\lambda} = \left(\frac{-40}{94.3}\right)\vec{i} + \left(\frac{80}{94.3}\right)\vec{j} + \left(\frac{30}{94.3}\right)\vec{k} = -0.424\vec{i} + 0.848\vec{j} + 0.318\vec{k}$$

- Determine the components of the force.

$$\vec{F} = F\vec{\lambda} = (2500\,\text{N})\left(-0.424\vec{i} + 0.848\vec{j} + 0.318\vec{k}\right) = (-1060\,\text{N})\vec{i} + (2120\,\text{N})\vec{j} + (795\,\text{N})\vec{k}$$

Sample Problem 2.7



- Noting that the components of the unit vector are the direction cosines for the vector, calculate the corresponding angles.

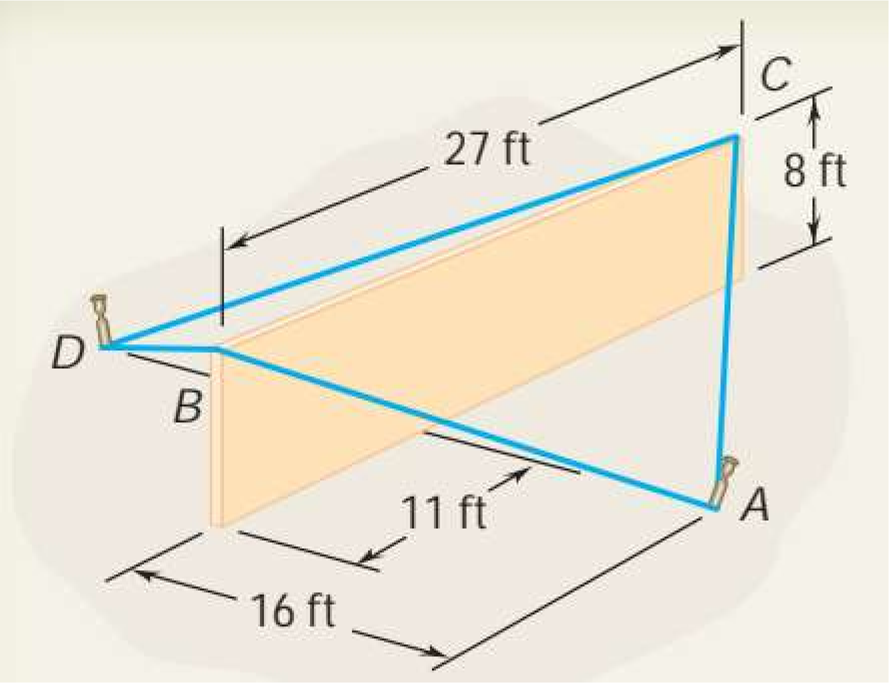
$$\vec{\lambda} = \cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}$$

$$= -0.424\vec{i} + 0.848\vec{j} + 0.318\vec{k}$$

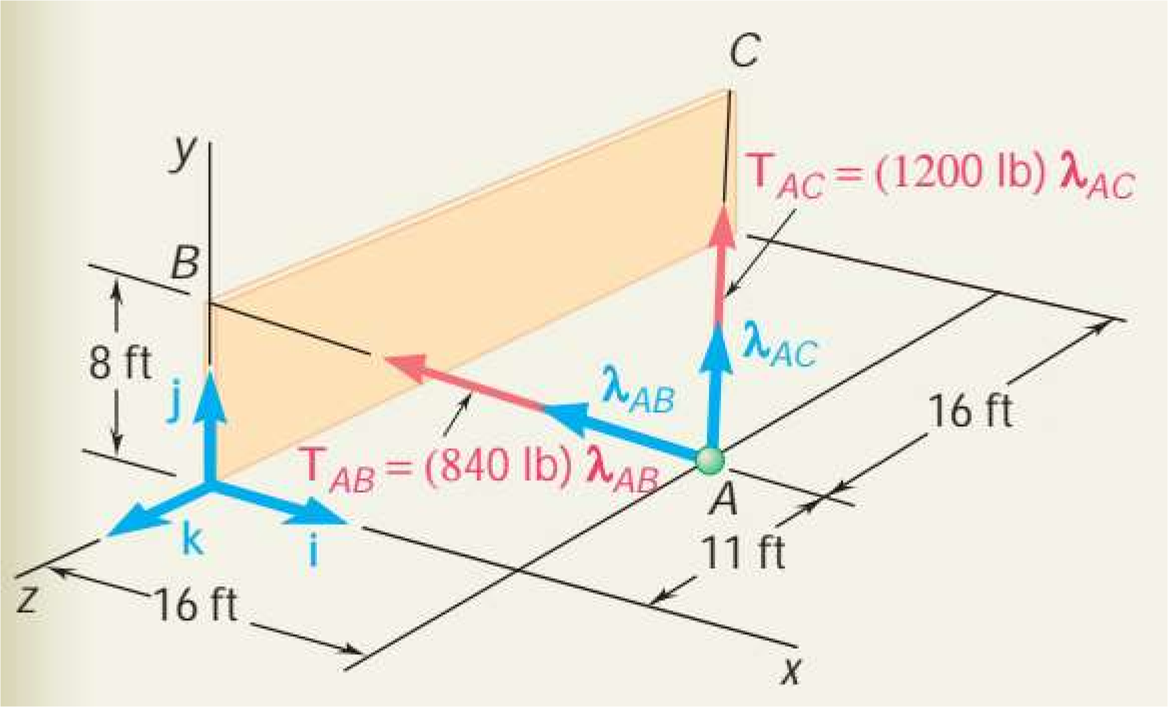
$$\boxed{\begin{aligned} \theta_x &= 115.1^\circ \\ \theta_y &= 32.0^\circ \\ \theta_z &= 71.5^\circ \end{aligned}}$$

Sample Problem 2.8

A wall section of precast concrete is temporarily held by the cables shown. Knowing that the tension is 840 lb in cable *AB* and 1200 lb in cable *AC*, determine the magnitude and direction of the resultant of the forces exerted by cables *AB* and *AC* on stake *A*.



Sample Problem 2.8



Sample Problem 2.8

Components of the Forces. The force exerted by each cable on stake A will be resolved into x, y, and z components. We first determine the components and magnitude of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, measuring them from A toward the wall section. Denoting by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along the coordinate axes, we write

$$\overrightarrow{AB} = -(16\text{ ft})\mathbf{i} + (8\text{ ft})\mathbf{j} + (11\text{ ft})\mathbf{k} \qquad AB = 21\text{ ft}$$
$$\overrightarrow{AC} = -(16\text{ ft})\mathbf{i} + (8\text{ ft})\mathbf{j} - (16\text{ ft})\mathbf{k} \qquad AC = 24\text{ ft}$$

Denoting by $\boldsymbol{\lambda}_{AB}$ the unit vector along AB, we have

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = \frac{840\text{ lb}}{21\text{ ft}}\overrightarrow{AB}$$

Sample Problem 2.8

Substituting the expression found for $\overrightarrow{AB}$, we obtain

$$\mathbf{T}_{AB} = \frac{840 \text{ lb}}{21 \text{ ft}}[-(16 \text{ ft})\mathbf{i} + (8 \text{ ft})\mathbf{j} + (11 \text{ ft})\mathbf{k}]$$

$$\mathbf{T}_{AB} = -(640 \text{ lb})\mathbf{i} + (320 \text{ lb})\mathbf{j} + (440 \text{ lb})\mathbf{k}$$

Denoting by $\boldsymbol{\lambda}_{AC}$ the unit vector along AC, we obtain in a similar way

$$\mathbf{T}_{AC} = T_{AC}\lambda_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = \frac{1200 \text{ lb}}{24 \text{ ft}}\overrightarrow{AC}$$

$$\mathbf{T}_{AC} = -(800 \text{ lb})\mathbf{i} + (400 \text{ lb})\mathbf{j} - (800 \text{ lb})\mathbf{k}$$

Sample Problem 2.8

Resultant of the Forces. The resultant **R** of the forces exerted by the two cables is

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} = -(1440 \text{ lb})\mathbf{i} + (720 \text{ lb})\mathbf{j} - (360 \text{ lb})\mathbf{k}$$

The magnitude and direction of the resultant are now determined:

$$R = 2\overline{R_x^2 + R_y^2 + R_z^2} = 2\overline{(-1440)^2 + (720)^2 + (-360)^2}$$

$$R = 1650 \text{ lb}$$

From Eqs. (2.33) we obtain

$$\cos u_x = \frac{R_x}{R} = \frac{-1440 \text{ lb}}{1650 \text{ lb}} \qquad \cos u_y = \frac{R_y}{R} = \frac{+720 \text{ lb}}{1650 \text{ lb}}$$

$$\cos u_z = \frac{R_z}{R} = \frac{-360 \text{ lb}}{1650 \text{ lb}}$$

Calculating successively each quotient and its arc cosine, we have

$$u_x = 150.8° \qquad u_y = 64.1° \qquad u_z = 102.6°$$

Equilibrium in Space

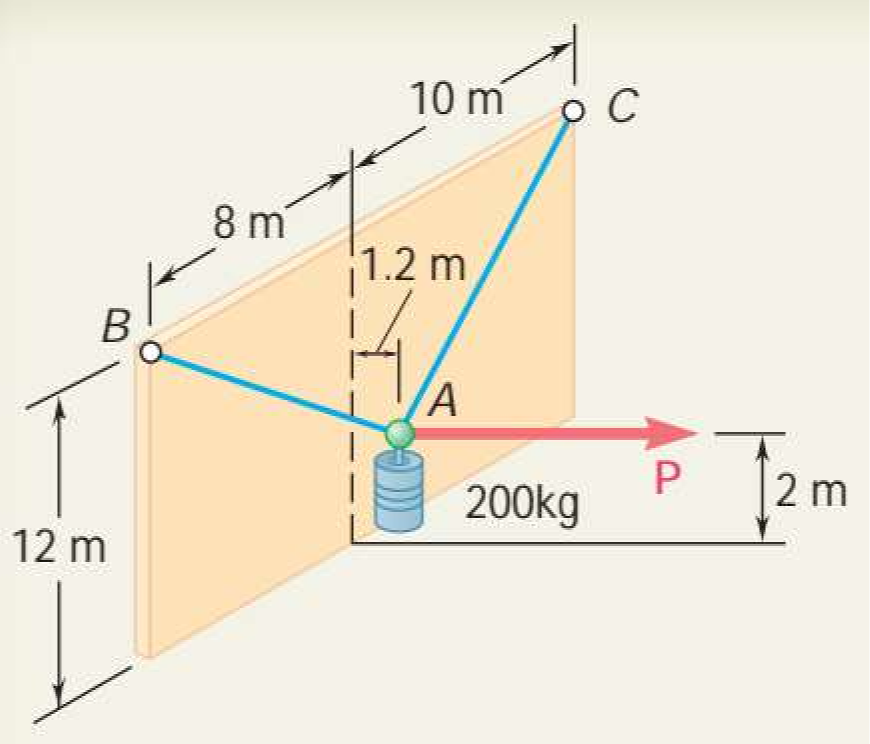
Equilibrium of a particle in space

If an object in is equilibrium and if the problem is represented in three dimensions, then the relationship ΣF = 0 can be expressed as:

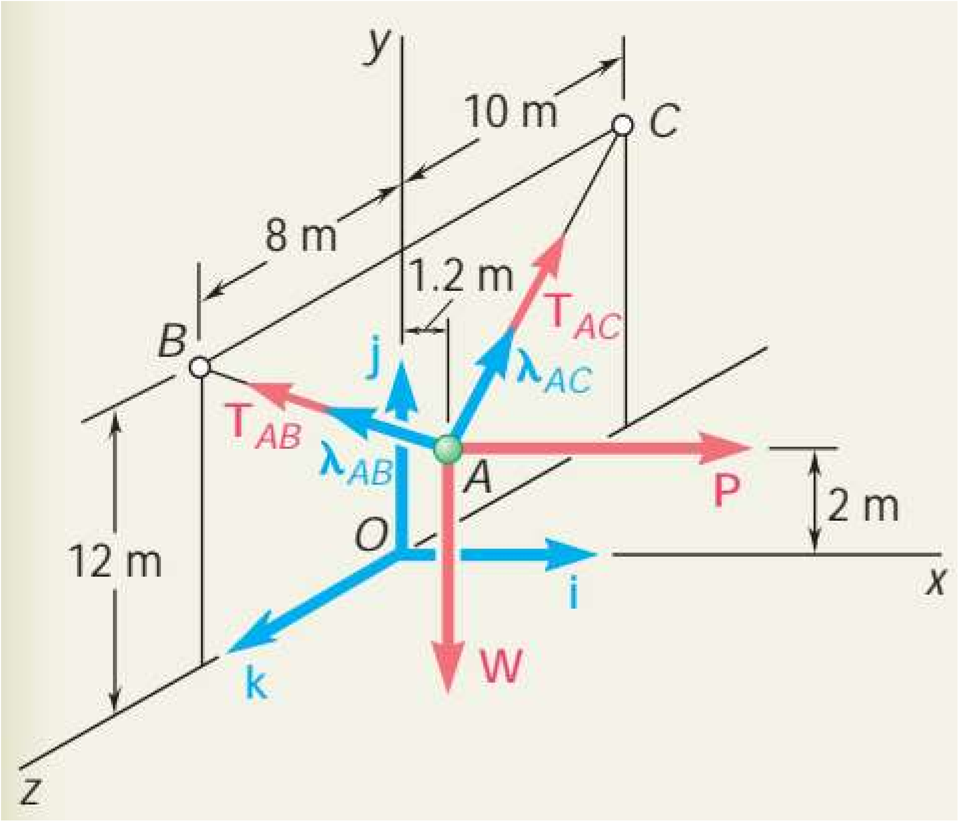
$$\boxed{\begin{aligned}\Sigma F_x &= 0\\ \Sigma F_y &= 0\\ \Sigma F_z &= 0\end{aligned}}$$

Sample Problem 2.9

A 200-kg cylinder is hung by means of two cables *AB* and *AC*, which are attached to the top of a vertical wall. A horizontal force **P** perpendicular to the wall holds the cylinder in the position shown. Determine the magnitude of **P** and the tension in each cable.



Sample Problem 2.9



Sample Problem 2.9

Free-Body Diagram. Point A is chosen as a free body; this point is subjected to four forces, three of which are of unknown magnitude.

Introducing the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we resolve each force into rectangular components.

$$\mathbf{P} = P\mathbf{i}$$
$$\mathbf{W} = -mg\mathbf{j} = -(200 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j} = -(1962 \text{ N})\mathbf{j} \quad (1)$$

In the case of $\mathbf{T}_{AB}$ and $\mathbf{T}_{AC}$, it is necessary first to determine the components and magnitudes of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Denoting by λ_{AB} the unit vector along AB, we write

$$\overrightarrow{AB} = -(1.2 \text{ m})\mathbf{i} + (10 \text{ m})\mathbf{j} + (8 \text{ m})\mathbf{k} \qquad AB = 12.862 \text{ m}$$
$$\lambda_{AB} = \frac{\overrightarrow{AB}}{12.862 \text{ m}} = -0.09330\mathbf{i} + 0.7775\mathbf{j} + 0.6220\mathbf{k}$$

Sample Problem 2.9

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = -0.09330T_{AB}\mathbf{i} + 0.7775T_{AB}\mathbf{j} + 0.6220T_{AB}\mathbf{k} \qquad (2)$$

Denoting by $\boldsymbol{\lambda}_{AC}$ the unit vector along AC, we write in a similar way

$$\overrightarrow{AC} = -(1.2 \text{ m})\mathbf{i} + (10 \text{ m})\mathbf{j} - (10 \text{ m})\mathbf{k} \qquad AC = 14.193 \text{ m}$$

$$\boldsymbol{\lambda}_{AC} = \frac{\overrightarrow{AC}}{14.193 \text{ m}} = -0.08455\mathbf{i} + 0.7046\mathbf{j} - 0.7046\mathbf{k}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = -0.08455T_{AC}\mathbf{i} + 0.7046T_{AC}\mathbf{j} - 0.7046T_{AC}\mathbf{k} \qquad (3)$$

Sample Problem 2.9

Equilibrium Condition. Since A is in equilibrium, we must have

$$\Sigma \mathbf{F} = 0: \qquad \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{P} + \mathbf{W} = 0$$

or, substituting from (1), (2), (3) for the forces and factoring **i**, **j**, **k**,

$$\begin{aligned}(-0.09330T_{AB} - 0.08455T_{AC} + P)\mathbf{i} \\ + (0.7775T_{AB} + 0.7046T_{AC} - 1962\ \text{N})\mathbf{j} \\ + (0.6220T_{AB} - 0.7046T_{AC})\mathbf{k} = 0\end{aligned}$$

Setting the coefficients of **i**, **j**, **k** equal to zero, we write three scalar equations, which express that the sums of the x, y, and z components of the forces are respectively equal to zero.

$$(\Sigma F_x = 0:) \qquad -0.09330T_{AB} - 0.08455T_{AC} + P = 0$$
$$(\Sigma F_y = 0:) \qquad +0.7775T_{AB} + 0.7046T_{AC} - 1962\ \text{N} = 0$$
$$(\Sigma F_z = 0:) \qquad +0.6220T_{AB} - 0.7046T_{AC} = 0$$

Solving these equations, we obtain

$$P = 235\ \text{N} \qquad T_{AB} = 1402\ \text{N} \qquad T_{AC} = 1238\ \text{N}$$

CHAPTER

3

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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Static of Rigid Bodies:

1. Equivalent Systems of Forces
2. Equilibrium of Rigid Bodies

Contents

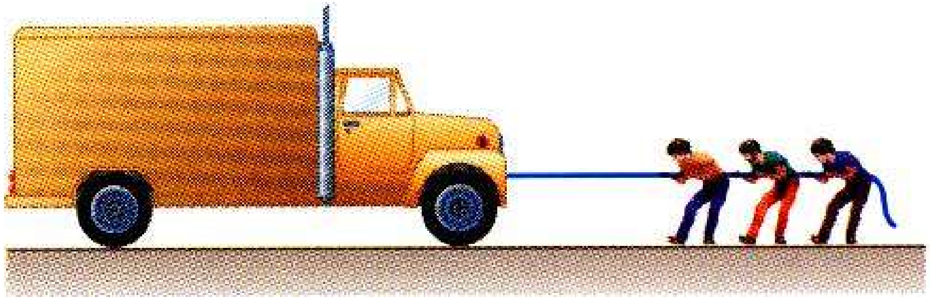
Contents

Introduction

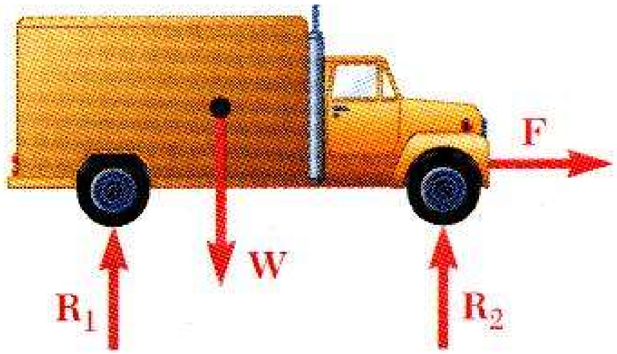
- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.
- Most bodies in elementary mechanics are assumed to be rigid, i.e., the actual deformations are small and do not affect the conditions of equilibrium or motion of the body.
- Current chapter describes the effect of forces exerted on a rigid body and how to replace a given system of forces with a simpler equivalent system.
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple
- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

External and Internal Forces

- Forces acting on rigid bodies are divided into two groups:
  - External forces
  - Internal forces



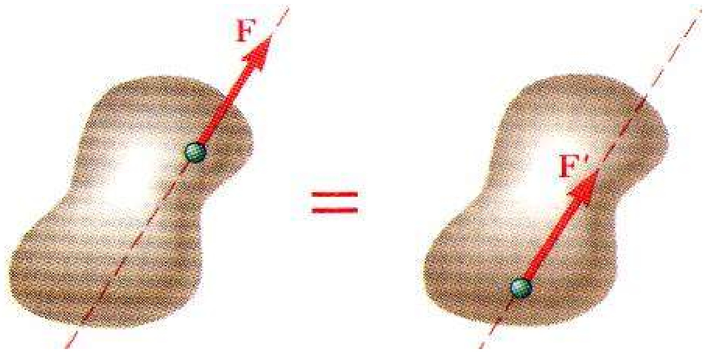
- External forces are shown in a free-body diagram.



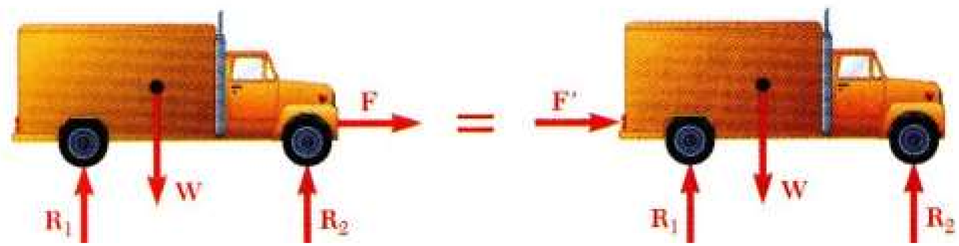
- If unopposed, each external force can impart a motion of translation or rotation, or both.

Principle of Transmissibility: Equivalent Forces

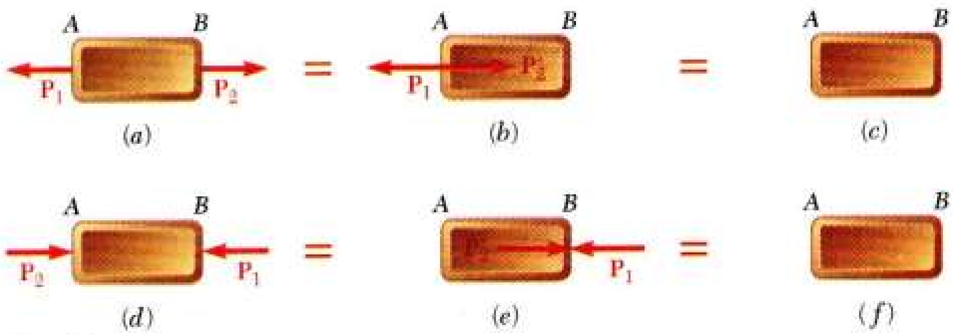
- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
  NOTE: **F** and **F'** are equivalent forces.



- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



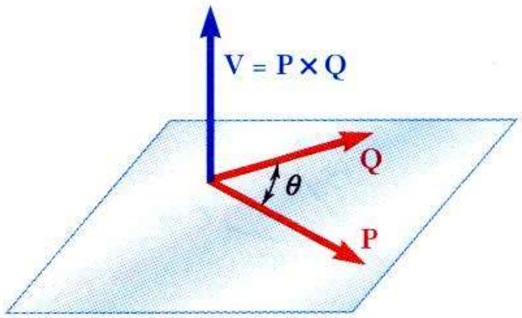
- Principle of transmissibility may not always apply in determining internal forces and deformations.



Vector Product of Two Vectors

- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
    1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
    2. Magnitude of $\boldsymbol{V}$ is $V = PQ \sin\theta$
    3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.



(a)

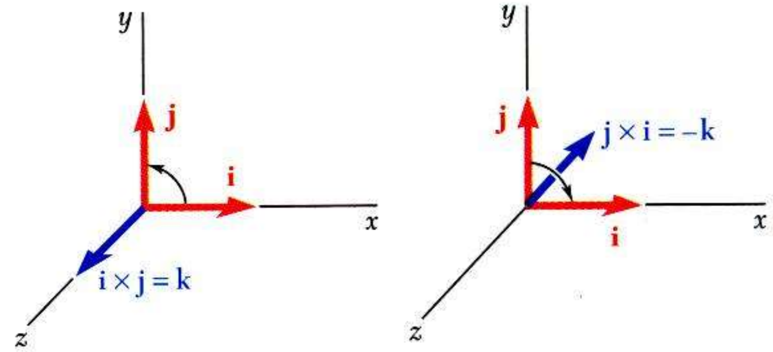
(b)

- Vector products:
    - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
    - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
    - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Vector Products: Rectangular Components

- Vector products of Cartesian unit vectors,

$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$

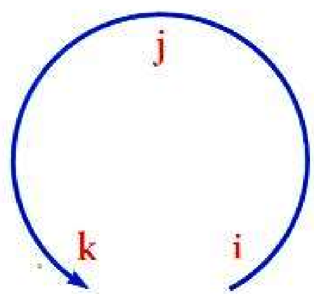


- Vector products in terms of rectangular coordinates

$$\vec{V}=\left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$

$$=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}$$
$$+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$



Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on it point of application.

- The *moment* of $\boldsymbol{F}$ about O is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

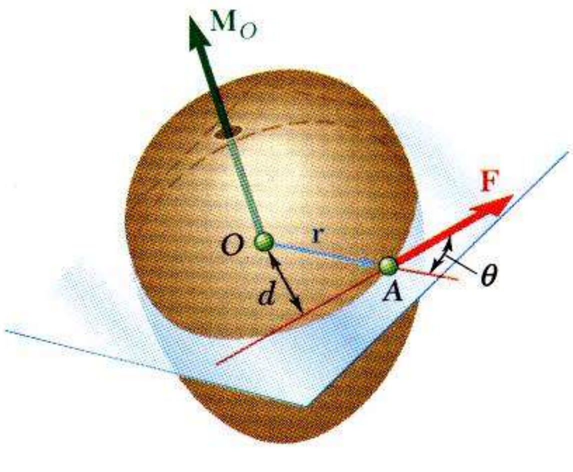
- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing O and the force $\boldsymbol{F}$.

- Magnitude of $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M_O}$.

$$M_O = rF \sin\theta = Fd$$

  The sense of the moment may be determined by the right-hand rule.

- Any force $\boldsymbol{F'}$ that has the same magnitude and direction as $\boldsymbol{F}$, is *equivalent* if it also has the same line of action and therefore, produces the same moment.



(a)

(b)

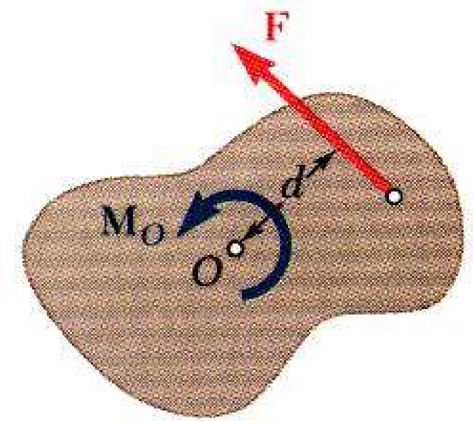
Moment of a Force About a Point

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained in the plane of the structure.

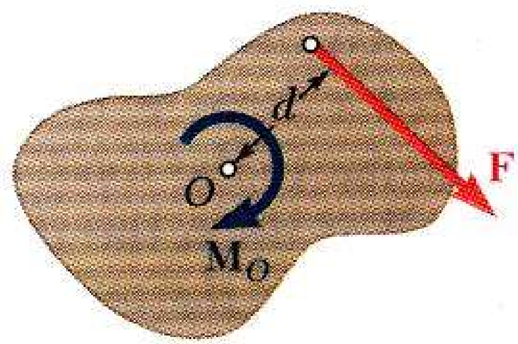
- The plane of the structure contains the point O and the force $\boldsymbol{F}$. $\boldsymbol{M_O}$, the moment of the force about O is perpendicular to the plane.

- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.

- If the force tends to rotate the structure clockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(a) $\boldsymbol{M}_O = + Fd$

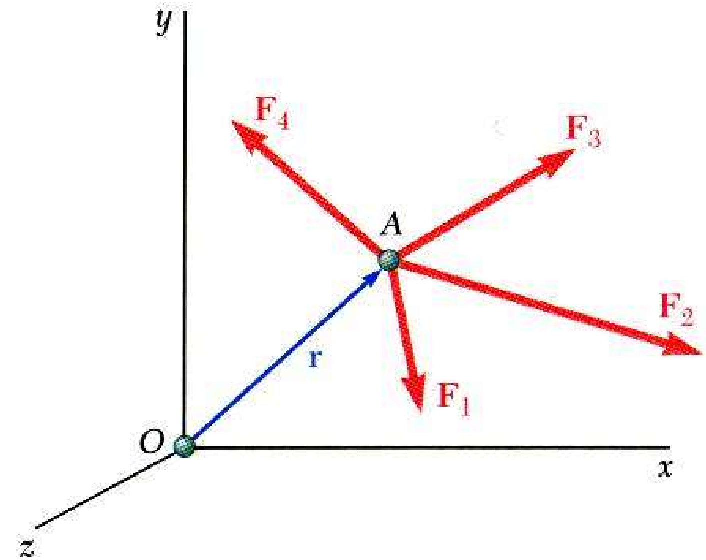


(b) $\boldsymbol{M}_O = - Fd$

Varignon's Theorem

- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

$$\vec{r}\times\left(\vec{F}_1+\vec{F}_2+\cdots\right)=\vec{r}\times\vec{F}_1+\vec{r}\times\vec{F}_2+\cdots$$



- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

Rectangular Components of the Moment of a Force

The moment of ***F*** about *O*,

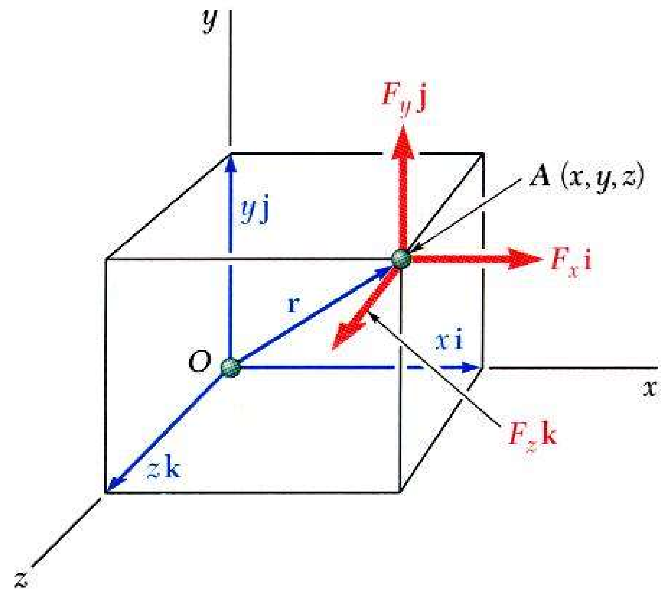
$$\vec{M}_O = \vec{r} \times \vec{F}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$



Rectangular Components of the Moment of a Force

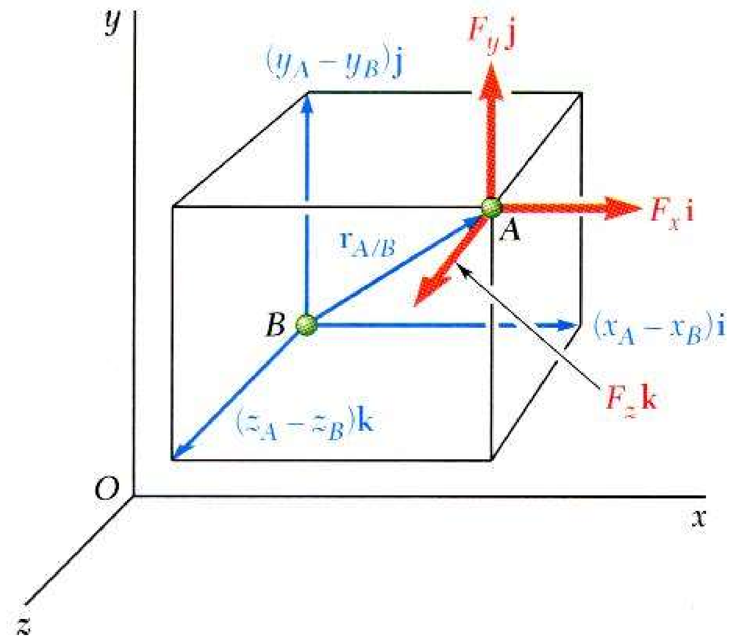
The moment of $\boldsymbol{F}$ about B,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

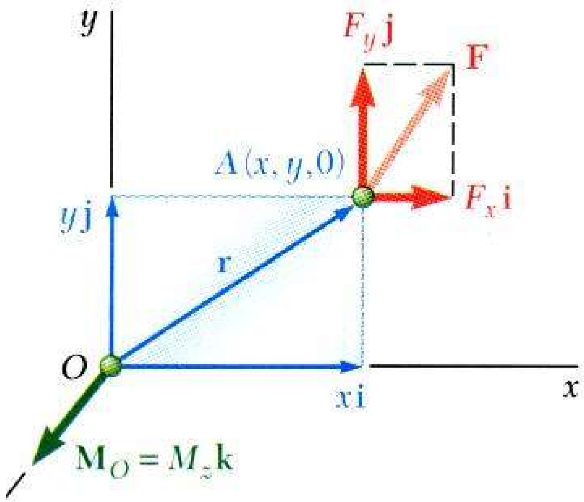
Rectangular Components of the Moment of a Force

For two-dimensional structures,

$$\vec{M}_O = \left(xF_y - yF_x\right)\vec{k}$$

$$M_O = M_Z$$

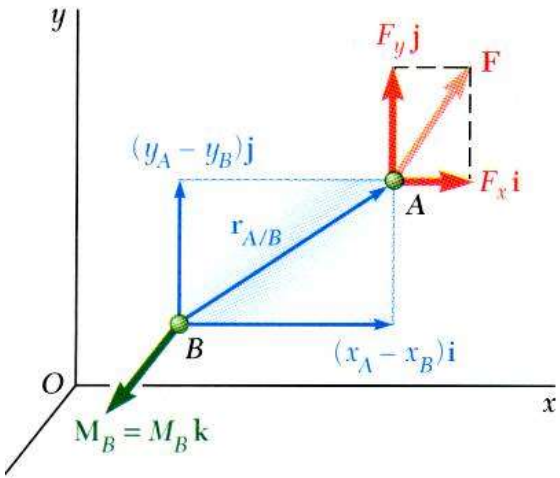
$$= xF_y - yF_x$$



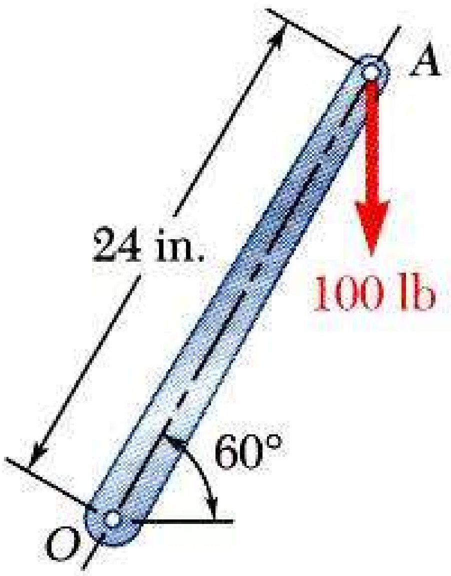
$$\vec{M}_O = \left[\left(x_A - x_B\right)F_y - \left(y_A - y_B\right)F_x\right]\vec{k}$$

$$M_O = M_Z$$

$$= \left(x_A - x_B\right)F_y - \left(y_A - y_B\right)F_x$$



Sample Problem 3.1



A 100lb vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

a) The moment of the 100lb force about *O*,

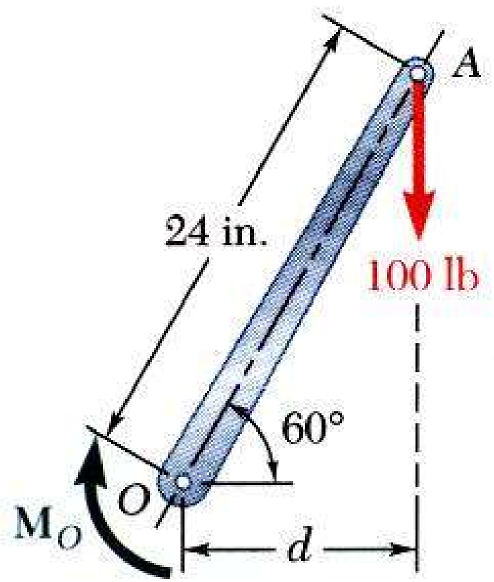
b) horizontal force at *A* which creates the same moment about *O*,

c) smallest force at A which produces the same moment about *O*,

d) location for a 240 lb vertical force to produce the same moment about *O*,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.1



a) Moment about O is equal to the product of the force and the perpendicular distance between the line of action of the force and O. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

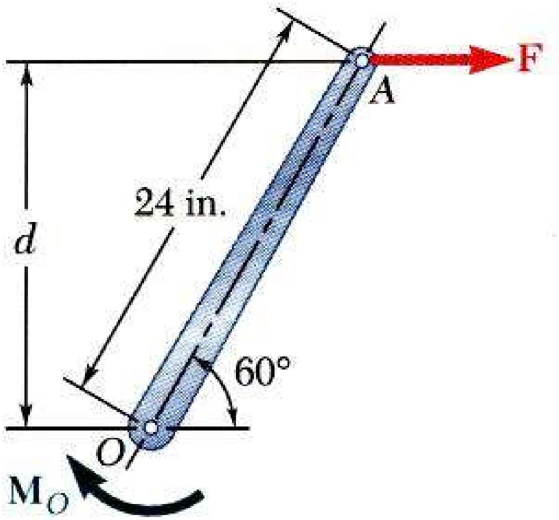
$$M_O = Fd$$

$$d = (24\text{in})\cos 60° = 12\text{ in.}$$

$$M_O = (100\text{ lb})(12\text{ in.})$$

$$\boxed{M_O = 1200\text{ lb.in.}}$$

Sample Problem 3.1



b) Horizontal force at A that produces the same moment,

$$d = (24 \text{ in.})\sin 60° = 20.8 \text{ in.}$$

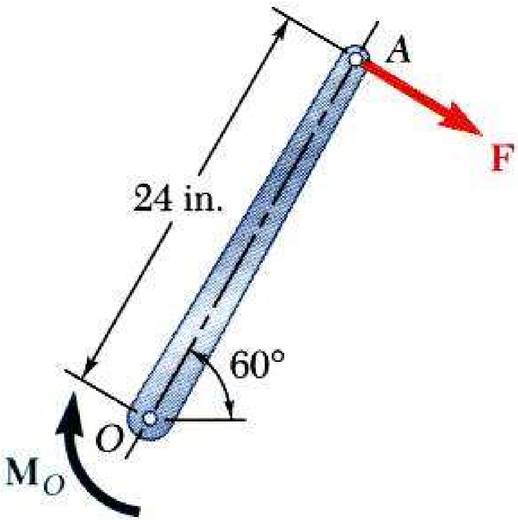
$$M_O = Fd$$

$$1200 \text{ lb} \cdot \text{in.} = F(20.8 \text{ in.})$$

$$F = \frac{1200 \text{ lb} \cdot \text{in.}}{20.8 \text{ in.}}$$

$$\boxed{F = 57.7 \text{lb}}$$

Sample Problem 3.1



c) The smallest force *A* to produce the same moment occurs when the perpendicular distance is a maximum or when *F* is perpendicular to *OA*.

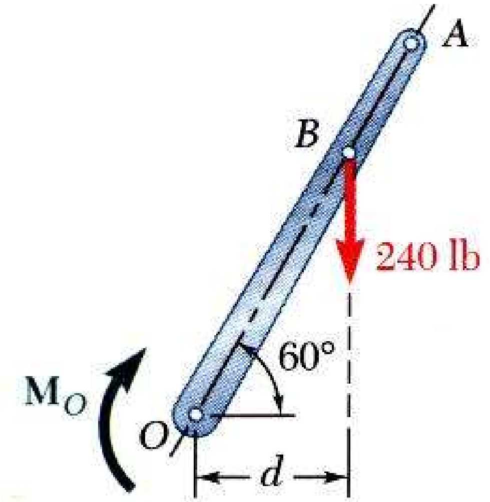
$$M_O = Fd$$

$$1200 \text{ lb} \cdot \text{in.} = F(24 \text{ in.})$$

$$F = \frac{1200 \text{ lb} \cdot \text{in.}}{24 \text{ in.}}$$

$$\boxed{F = 50 \text{ lb}}$$

Sample Problem 3.1



d) To determine the point of application of a 240 lb force to produce the same moment,

$$M_O = Fd$$

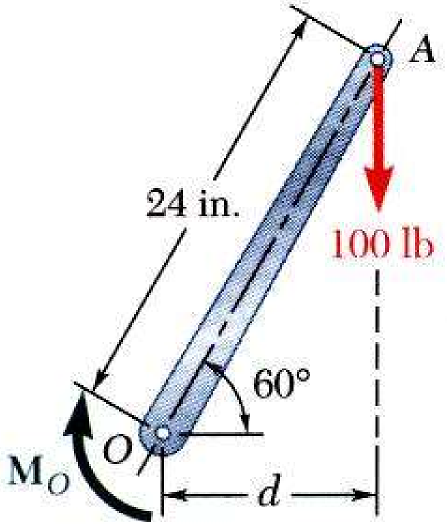
$$1200 \text{ lb}\cdot\text{in.} = (240 \text{ lb})d$$

$$d = \frac{1200 \text{ lb}\cdot\text{in.}}{240 \text{ lb}} = 5 \text{ in.}$$

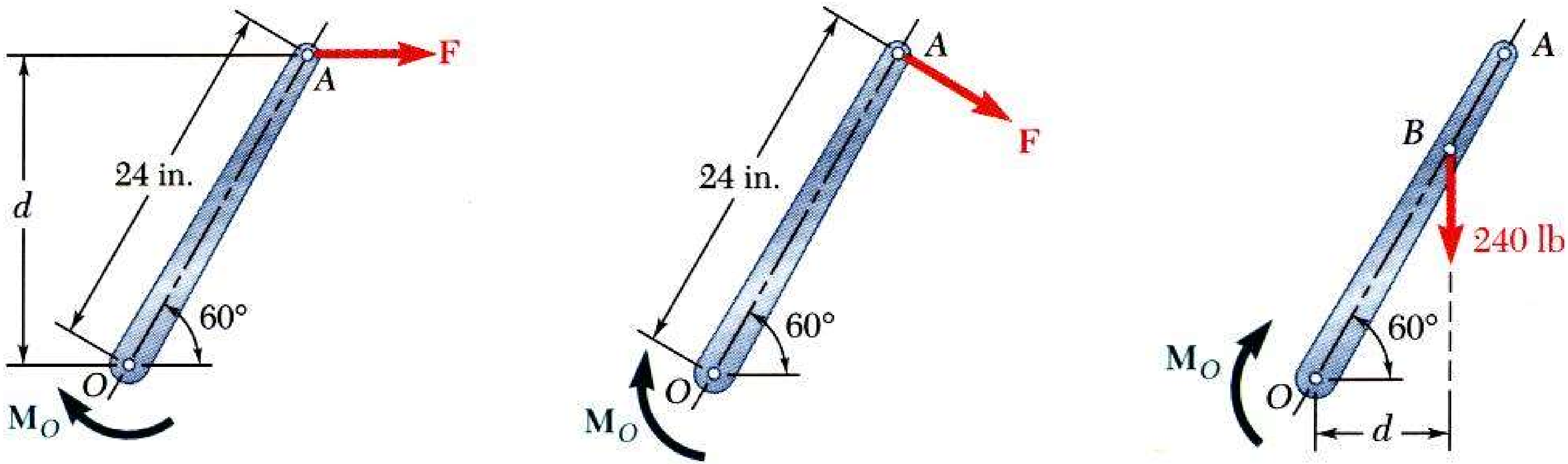
$$OB \cos 60° = 5 \text{ in.}$$

$$\boxed{OB = 10 \text{ in.}}$$

Sample Problem 3.1

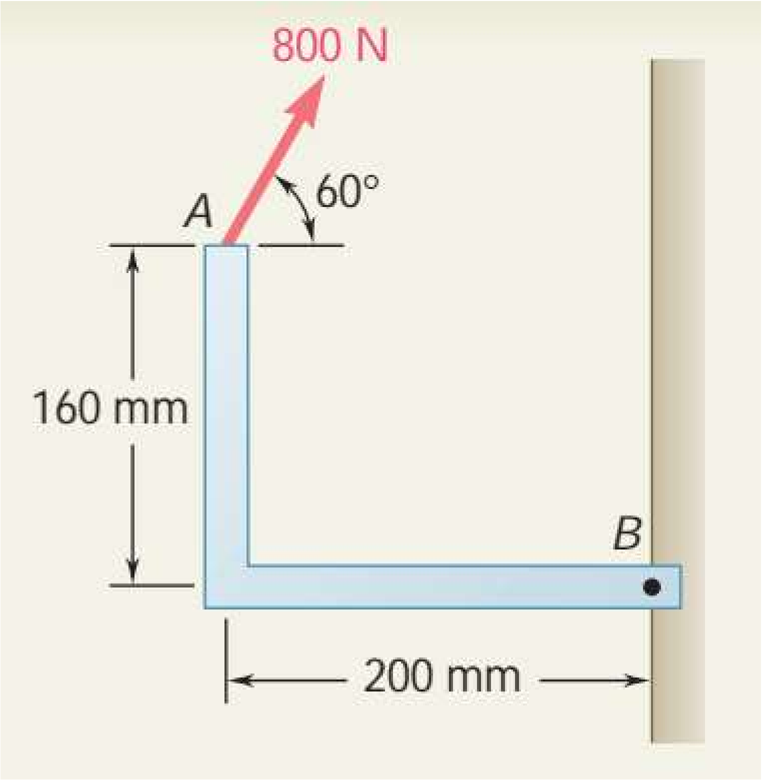


e) Although each of the forces in parts b), c), and d) produces the same moment as the 450 N force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 450 N force.



Sample Problem 3.2

A force of 800 N acts on a bracket as shown. Determine the moment of the force about B.



Sample Problem 3.2

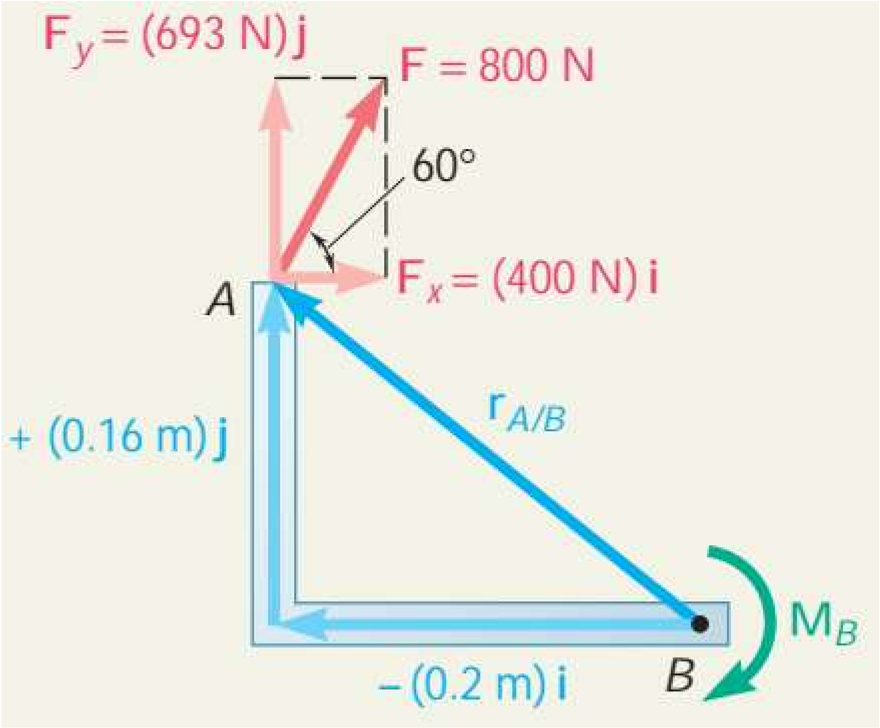
The moment $\mathbf{M}_B$ of the force $\mathbf{F}$ about B is obtained by forming the vector product

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F}$$

where $\mathbf{r}_{A/B}$ is the vector drawn from B to A. Resolving $\mathbf{r}_{A/B}$ and $\mathbf{F}$ into rectangular components, we have

$$\begin{aligned} \mathbf{r}_{A/B} &= -(0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j} \\ \mathbf{F} &= (800\text{ N})\cos 60°\mathbf{i} + (800\text{ N})\sin 60°\mathbf{j} \\ &= (400\text{ N})\mathbf{i} + (693\text{ N})\mathbf{j} \end{aligned}$$

Sample Problem 3.2



Sample Problem 3.2

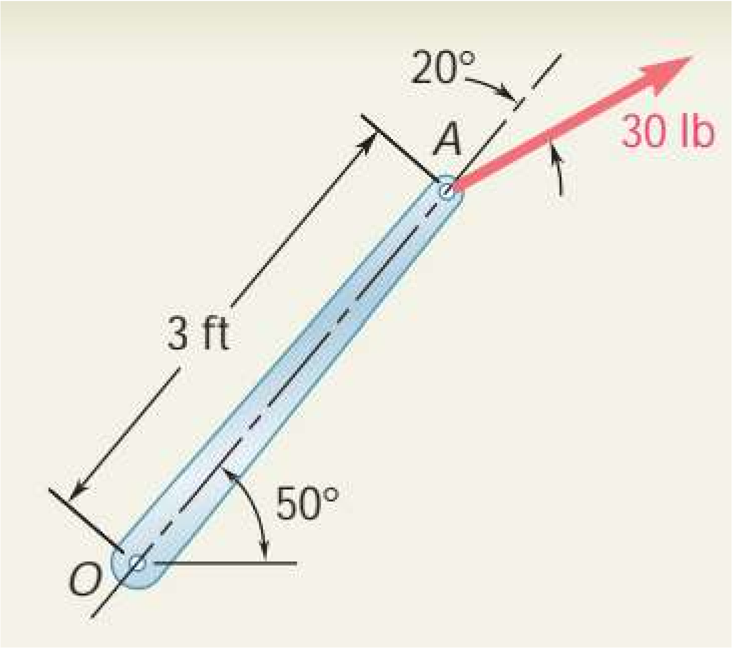
Recalling the relations (3.7) for the cross products of unit vectors (Sec. 3.5), we obtain

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} = [-(0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j}] \times [(400\text{ N})\mathbf{i} + (693\text{ N})\mathbf{j}]$$
$$= -(138.6\text{ N}\cdot\text{m})\mathbf{k} - (64.0\text{ N}\cdot\text{m})\mathbf{k}$$
$$= -(202.6\text{ N}\cdot\text{m})\mathbf{k} \qquad \mathbf{M}_B = 203\text{ N}\cdot\text{m i}$$

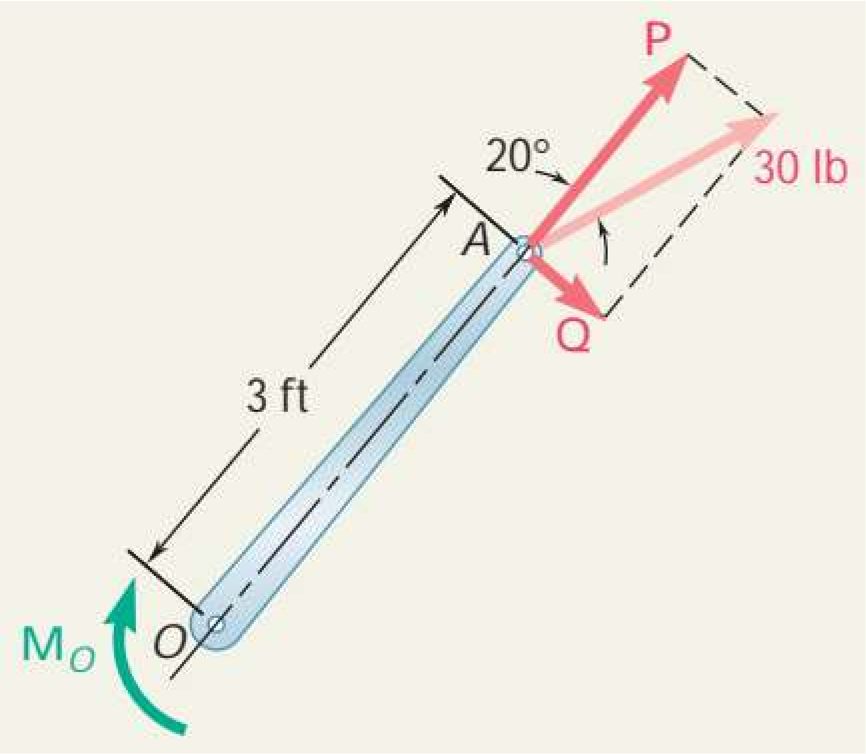
The moment $\mathbf{M}_B$ is a vector perpendicular to the plane of the figure and pointing *into* the paper.

Sample Problem 3.3

A 30-lb force acts on the end of the 3-ft lever as shown. Determine the moment of the force about O.



Sample Problem 3.3



Sample Problem 3.3

The force is replaced by two components, one component **P** in the direction of OA and one component **Q** perpendicular to OA. Since O is on the line of action of **P**, the moment of **P** about O is zero and the moment of the 30-lb force reduces to the moment of **Q**, which is clockwise and, thus, is represented by a negative scalar.

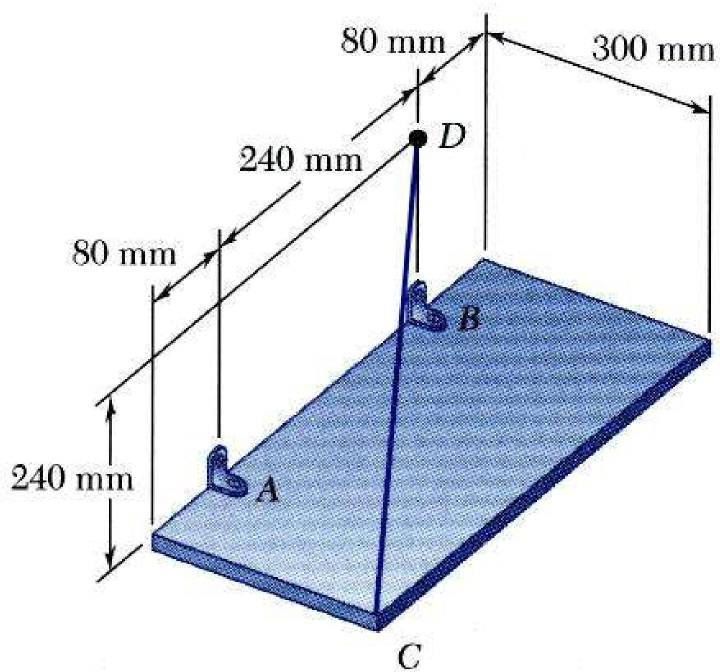
$$Q = (30 \text{ lb}) \sin 20° = 10.26 \text{ lb}$$

$$M_O = -Q(3 \text{ ft}) = -(10.26 \text{ lb})(3 \text{ ft}) = -30.8 \text{ lb} \cdot \text{ft}$$

Since the value obtained for the scalar M_O is negative, the moment $\mathbf{M}_O$ points *into* the paper. We write

$$\mathbf{M}_O = 30.8 \text{ lb} \cdot \text{ft i}$$

Sample Problem 3.4



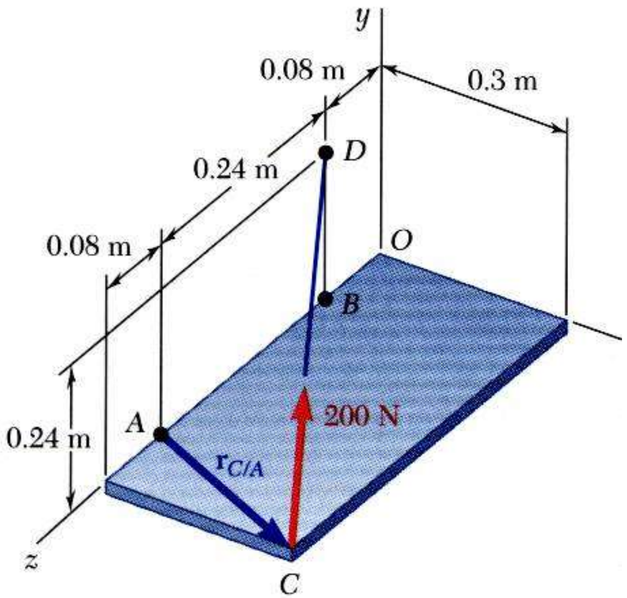
The rectangular plate is supported by the brackets at A and B and by a wire CD. Knowing that the tension in the wire is 200 N, determine the moment about A of the force exerted by the wire at C.

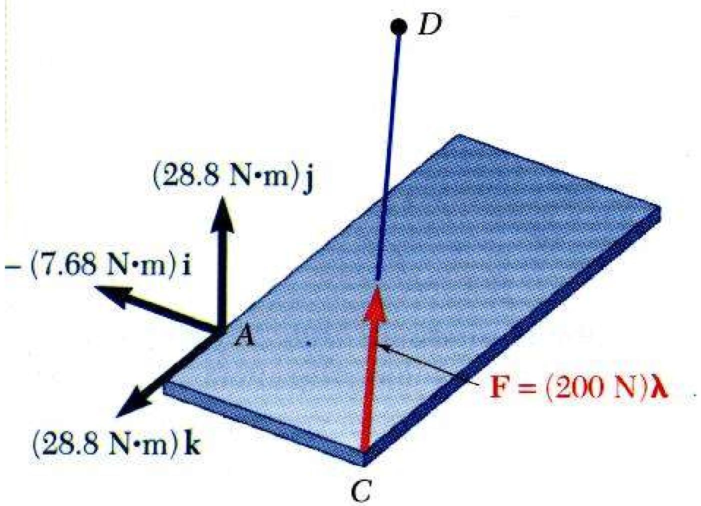
SOLUTION:

The moment M_A of the force F exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Sample Problem 3.4





SOLUTION:

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\ \text{m})\vec{i} + (0.08\ \text{m})\vec{k}$$

$$\vec{F} = F\vec{\lambda} = (200\ \text{N})\frac{\vec{r}_{C/D}}{r_{C/D}}$$

$$= (200\ \text{N})\frac{-(0.3\ \text{m})\vec{i} + (0.24\ \text{m})\vec{j} - (0.32\ \text{m})\vec{k}}{0.5\ \text{m}}$$

$$= -(120\ \text{N})\vec{i} + (96\ \text{N})\vec{j} - (128\ \text{N})\vec{k}$$

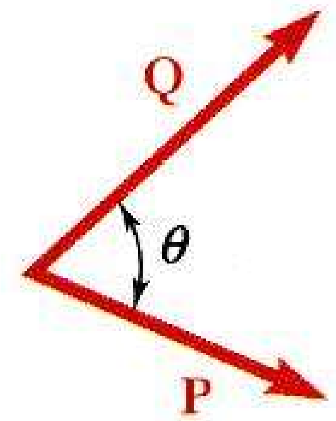
$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\vec{M}_A = -(7.68\ \text{N}\cdot\text{m})\vec{i} + (28.8\ \text{N}\cdot\text{m})\vec{j} + (28.8\ \text{N}\cdot\text{m})\vec{k}}$$

Scalar Product of Two Vectors

- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as

$$\vec{P}\bullet\vec{Q} = PQ\cos\theta \quad (\text{scalar result})$$



- Scalar products:
  - are commutative, $\vec{P}\bullet\vec{Q} = \vec{Q}\bullet\vec{P}$
  - are distributive, $\vec{P}\bullet\left(\vec{Q}_1 + \vec{Q}_2\right) = \vec{P}\bullet\vec{Q}_1 + \vec{P}\bullet\vec{Q}_2$
  - are not associative, $\left(\vec{P}\bullet\vec{Q}\right)\bullet\vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

$$\vec{P}\bullet\vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right)\bullet\left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$\vec{i}\bullet\vec{i} = 1 \quad \vec{j}\bullet\vec{j} = 1 \quad \vec{k}\bullet\vec{k} = 1 \quad \vec{i}\bullet\vec{j} = 0 \quad \vec{j}\bullet\vec{k} = 0 \quad \vec{k}\bullet\vec{i} = 0$$

$$\vec{P}\bullet\vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

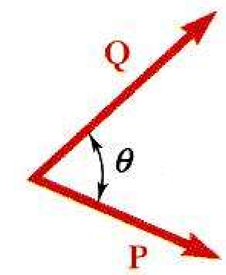
$$\vec{P}\bullet\vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Scalar Product of Two Vectors: Applications

- Angle between two vectors:

$$\vec{P}\bullet\vec{Q} = PQ\cos\theta = P_xQ_x + P_yQ_y + P_zQ_z$$

$$\cos\theta = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ}$$

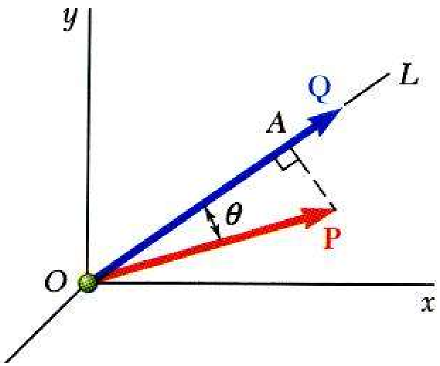


- Projection of a vector on a given axis:

$$P_{OL} = P\cos\theta = \text{ projection of } P \text{ along } OL$$

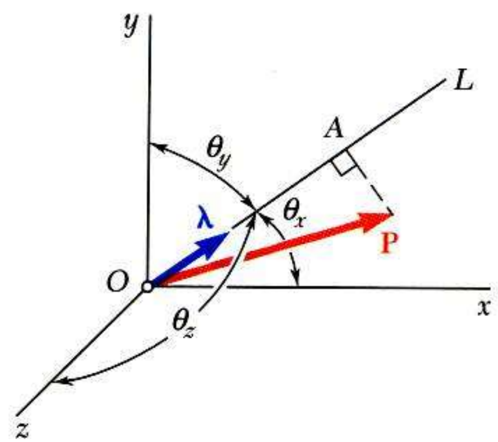
$$\vec{P}\bullet\vec{Q} = PQ\cos\theta = P_{OL}Q$$

$$\frac{\vec{P}\bullet\vec{Q}}{Q} = P\cos\theta = P_{OL} = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{Q}$$

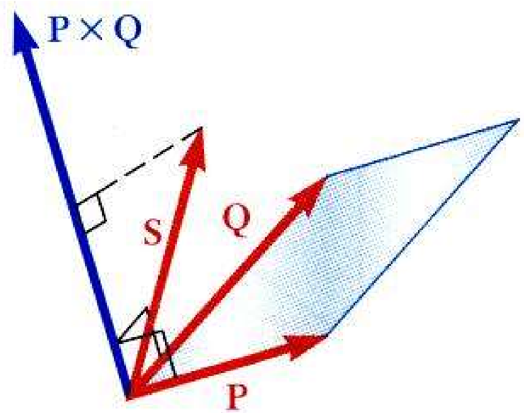


- For an axis defined by a unit vector:

$$P_{OL} = \vec{P}\bullet\vec{\lambda}$$
$$= P_x\cos\theta_x + P_y\cos\theta_y + P_z\cos\theta_z$$



Mixed Triple Product of Three Vectors



- Mixed triple product of three vectors,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \text{scalar result}$$

- The six mixed triple products formed from ***S***, ***P***, and ***Q*** have equal magnitudes but not the same sign,

$$\begin{aligned}\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) &= \vec{P} \bullet \left(\vec{Q} \times \vec{S}\right) = \vec{Q} \bullet \left(\vec{S} \times \vec{P}\right) \\ &= -\vec{S} \bullet \left(\vec{Q} \times P\right) = -\vec{P} \bullet \left(\vec{S} \times \vec{Q}\right) = -\vec{Q} \bullet \left(\vec{P} \times \vec{S}\right)\end{aligned}$$

- Evaluating the mixed triple product,

$$\begin{aligned}\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) &= S_x\left(P_y Q_z - P_z Q_y\right) + S_y\left(P_z Q_x - P_x Q_z\right) \\ &\quad + S_z\left(P_x Q_y - P_y Q_x\right) \\ &= \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}\end{aligned}$$

Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

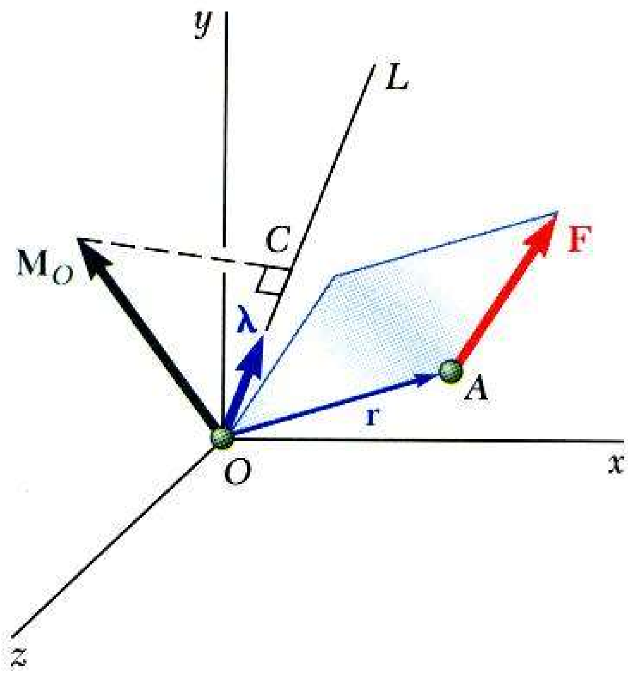
$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

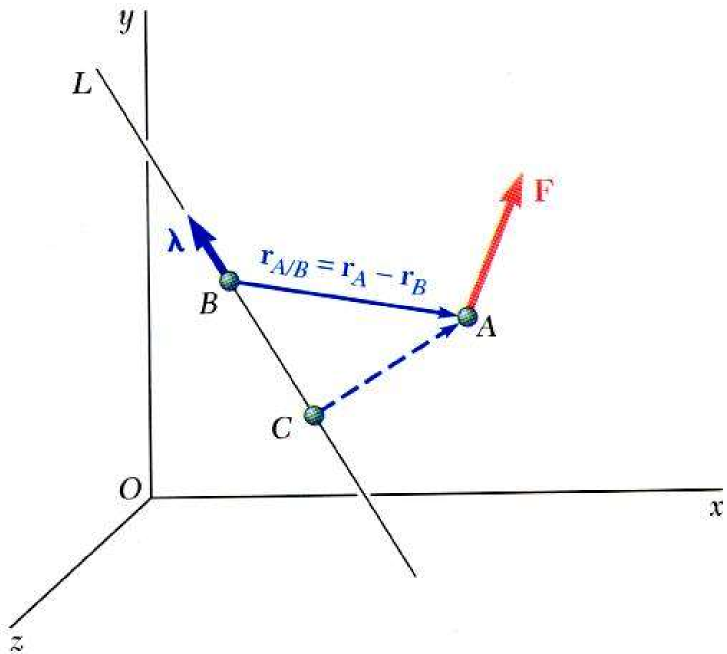
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

- Moments of $\boldsymbol{F}$ about the coordinate axes,

$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$



Moment of a Force About a Given Axis

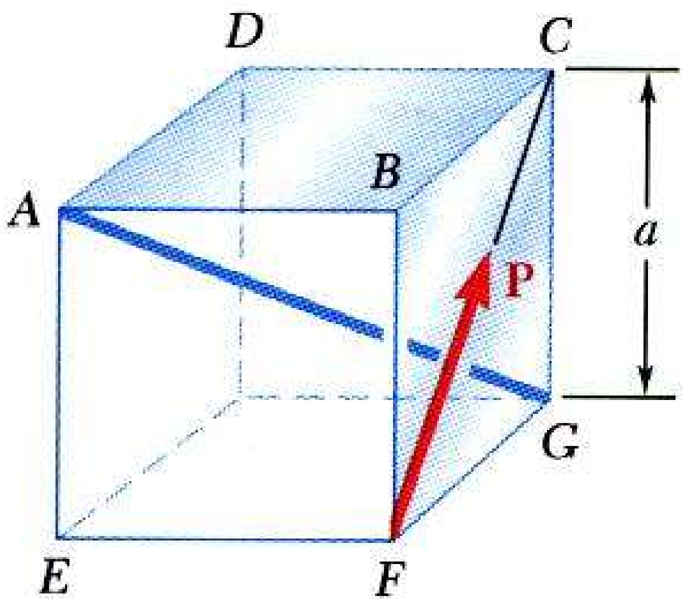


- Moment of a force about an arbitrary axis,

$$
\begin{aligned}
M_{BL} &= \vec{\lambda} \bullet \vec{M}_B \\
&= \vec{\lambda} \bullet \left(\vec{r}_{A/B} \times \vec{F} \right) \\
\vec{r}_{A/B} &= \vec{r}_A - \vec{r}_B
\end{aligned}
$$

- The result is independent of the point *B* along the given axis.

Sample Problem 3.5



A cube is acted on by a force ***P*** as shown. Determine the moment of ***P***

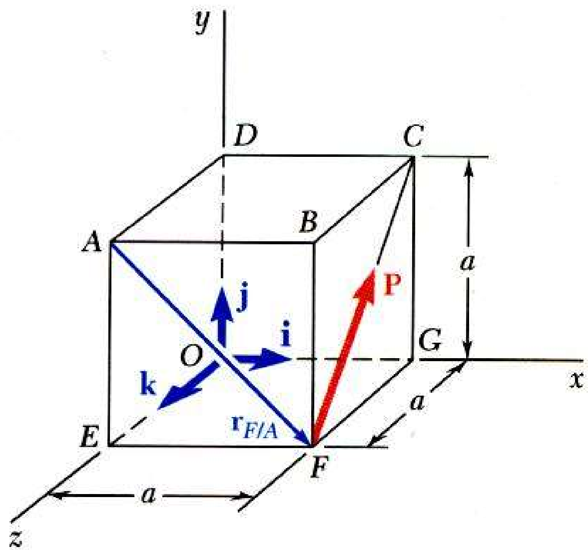
a) about *A*

b) about the edge *AB* and

c) about the diagonal *AG* of the cube.

d) Determine the perpendicular distance between *AG* and *FC*.

Sample Problem 3.5



- Moment of ***P*** about *A*,

$$\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$$

$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a\left(\vec{i} - \vec{j}\right)$$

$$\vec{P} = \frac{P\left(a\vec{j} - a\vec{k}\right)}{\sqrt{2}a} = \frac{P}{\sqrt{2}}\left(\vec{j} - \vec{k}\right)$$

$$\vec{M}_A = a\left(\vec{i} - \vec{j}\right) \times \frac{P}{\sqrt{2}}\left(\vec{j} - \vec{k}\right)$$

$$\boxed{\vec{M}_A = \left(\frac{aP}{\sqrt{2}}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)}$$

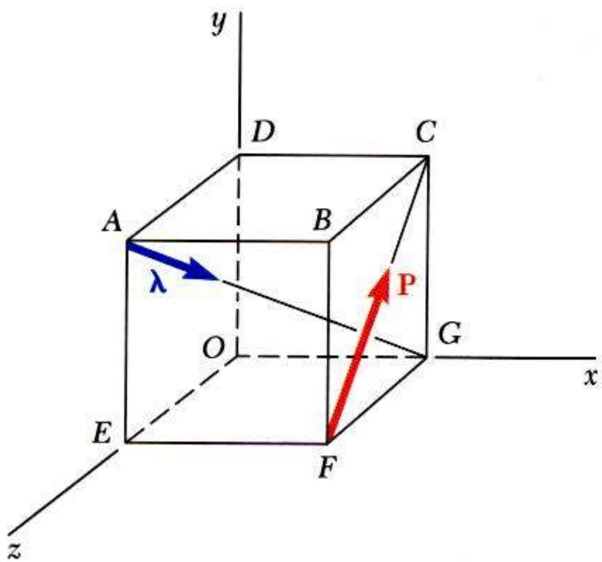
- Moment of ***P*** about *AB*,

$$M_{AB} = \vec{i} \bullet \vec{M}_A$$

$$= \vec{i} \bullet \left(aP/\sqrt{2} \right)\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$\boxed{M_{AB} = \frac{aP}{\sqrt{2}}}$$

Sample Problem 3.5



- Moment of ***P*** about the diagonal *AG*,

$$M_{AG} = \vec{\lambda} \bullet \vec{M}_A$$

$$\vec{\lambda} = \frac{\vec{r}_{A/G}}{r_{A/G}} = \frac{a\vec{i} - a\vec{j} - a\vec{k}}{a\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right)$$

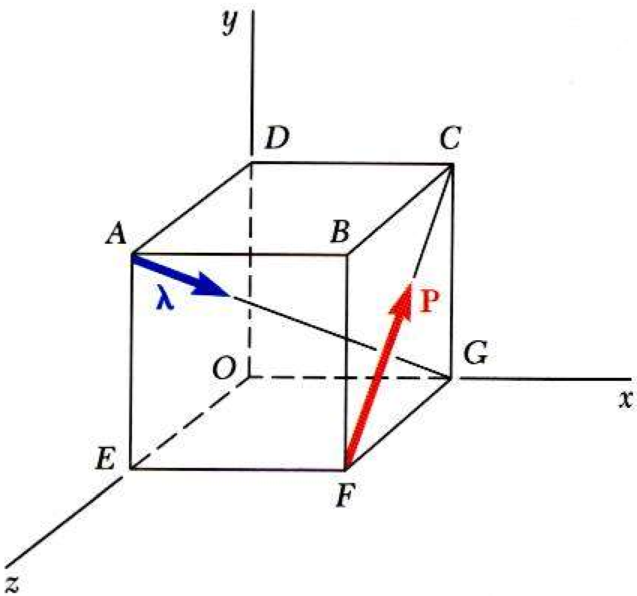
$$\vec{M}_A = \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

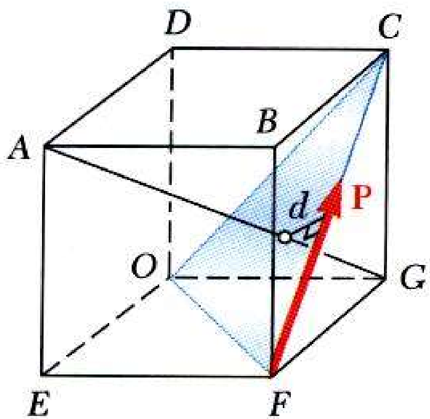
$$M_{AG} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) \bullet \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$= \frac{aP}{\sqrt{6}}(1 - 1 - 1)$$

$$\boxed{M_{AG} = -\frac{aP}{\sqrt{6}}}$$

Sample Problem 3.5





- Perpendicular distance between *AG* and *FC*,

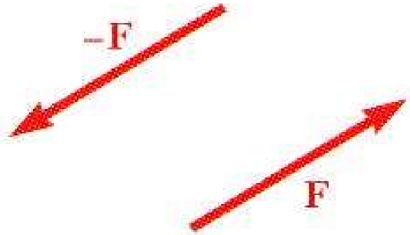
$$\vec{P}\bullet\vec{\lambda} = \frac{P}{\sqrt{2}}\left(\vec{j}-\vec{k}\right)\bullet\frac{1}{\sqrt{3}}\left(\vec{i}-\vec{j}-\vec{k}\right) = \frac{P}{\sqrt{6}}(0-1+1)$$
$$= 0$$

Therefore, *P* is perpendicular to *AG*.

$$\left|M_{AG}\right| = \frac{aP}{\sqrt{6}} = Pd$$

$$\boxed{d = \frac{a}{\sqrt{6}}}$$

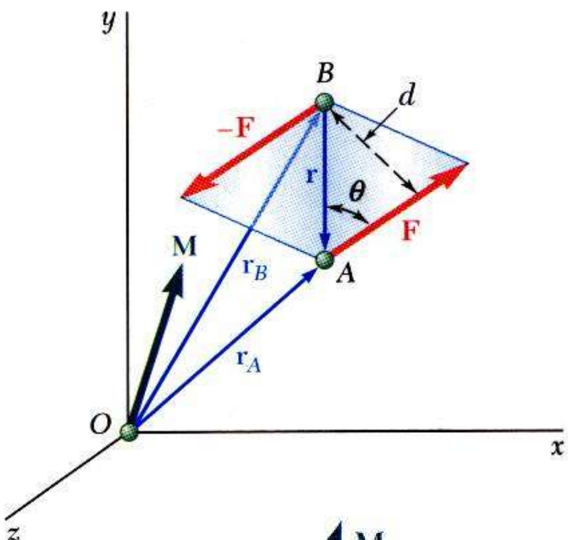
Moment of a Couple



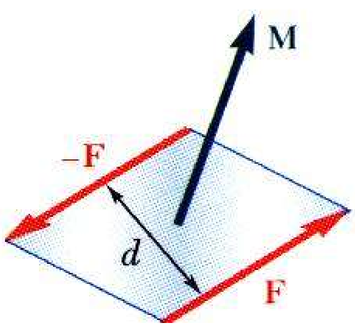
- Two forces ***F*** and -***F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

- Moment of the couple,

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF\sin\theta = Fd$$



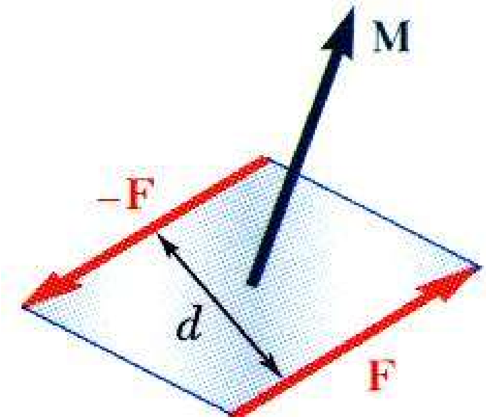
- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

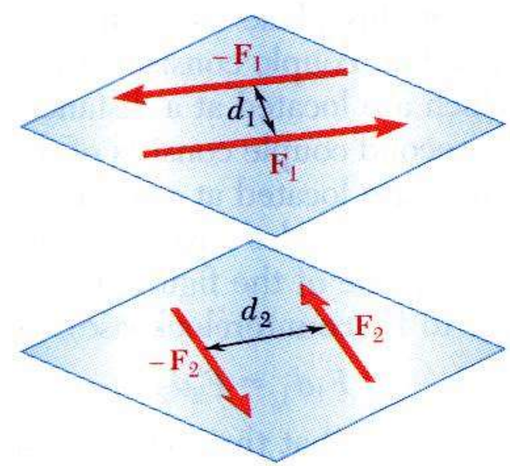


Moment of a Couple

Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.



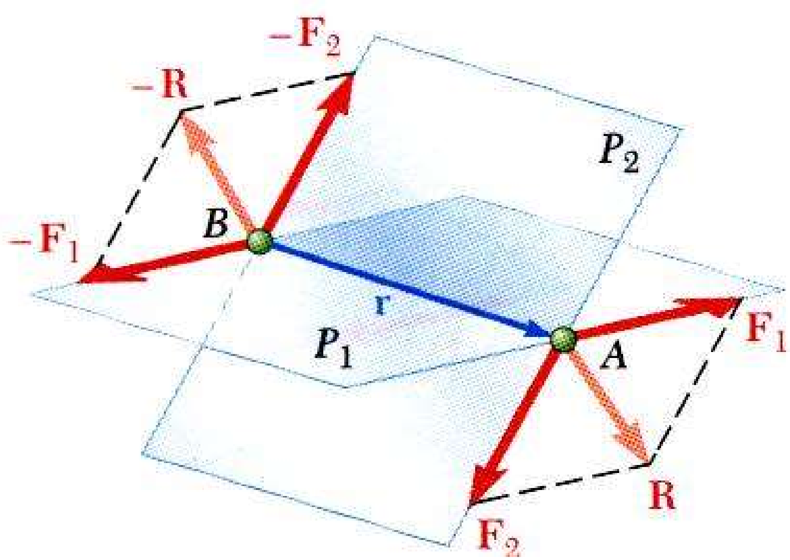


Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$

$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$



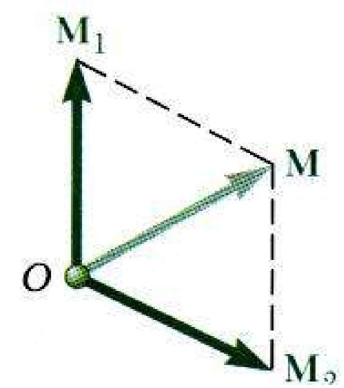
- Resultants of the vectors also form a couple

$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

- By Varigon's theorem

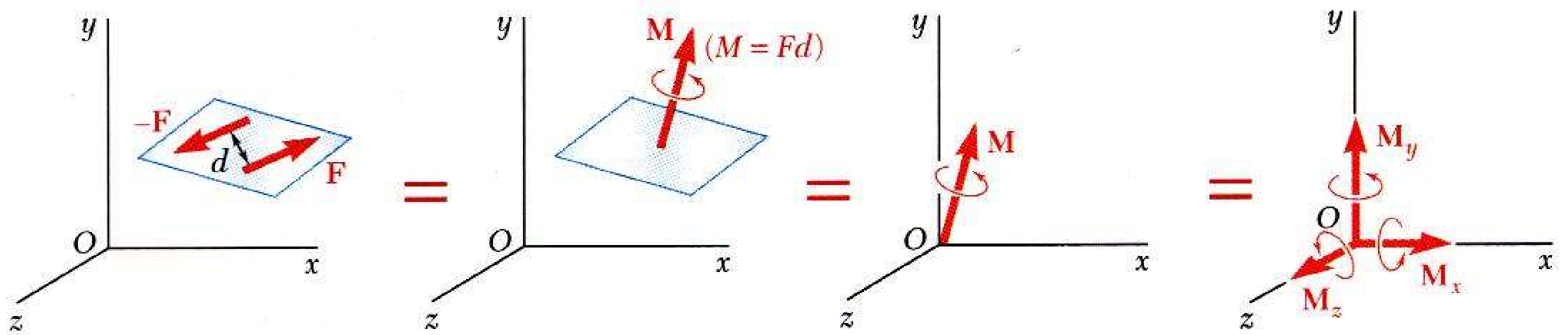
$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$

$$= \vec{M}_1 + \vec{M}_2$$



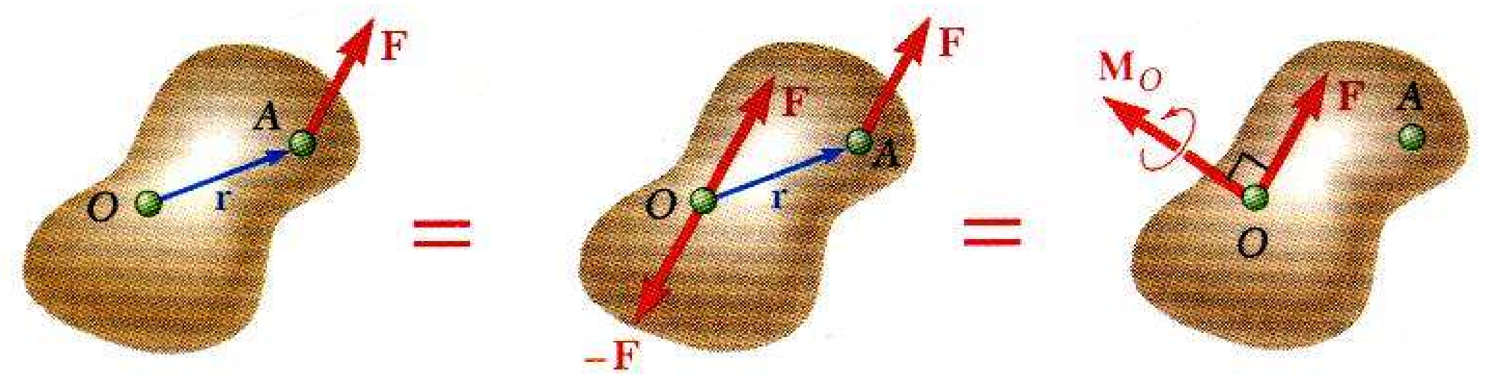
- Sum of two couples is also a couple that is equal to the vector sum of the two couples

Couples Can Be Represented by Vectors



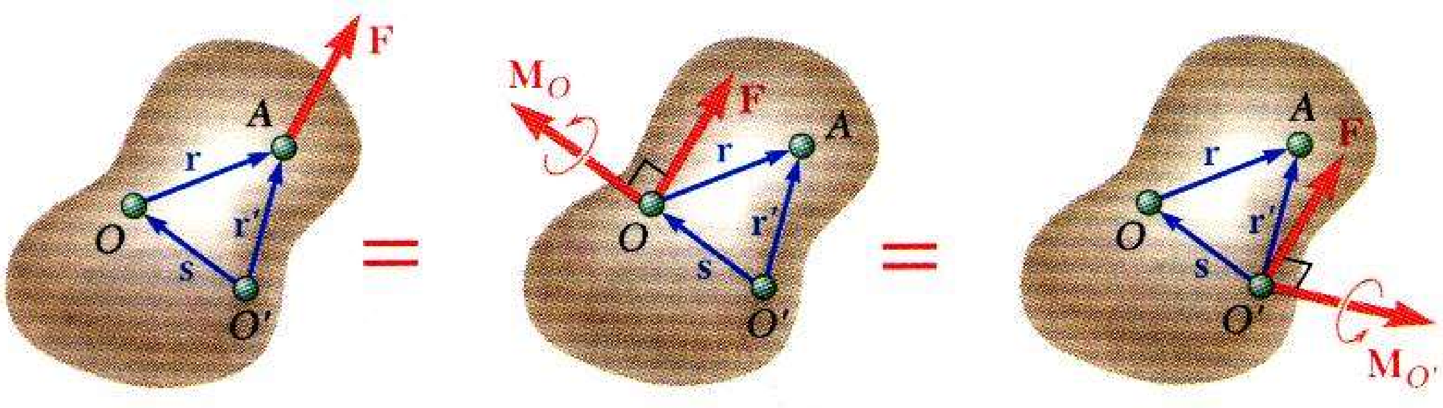
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force at O and a Couple



- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at O and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

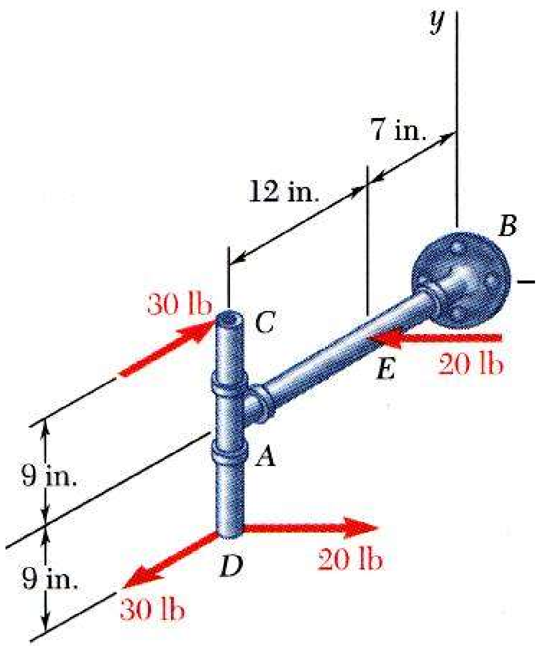
$$\vec{M}_{O'} = \vec{r}' \times \vec{F}$$

- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}' \times \vec{F} = (\vec{r} + \vec{s}) \times \vec{F} = \vec{r} \times \vec{F} + \vec{s} \times \vec{F}$$
$$= \vec{M}_O + \vec{s} \times \vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

Sample Problem 3.6



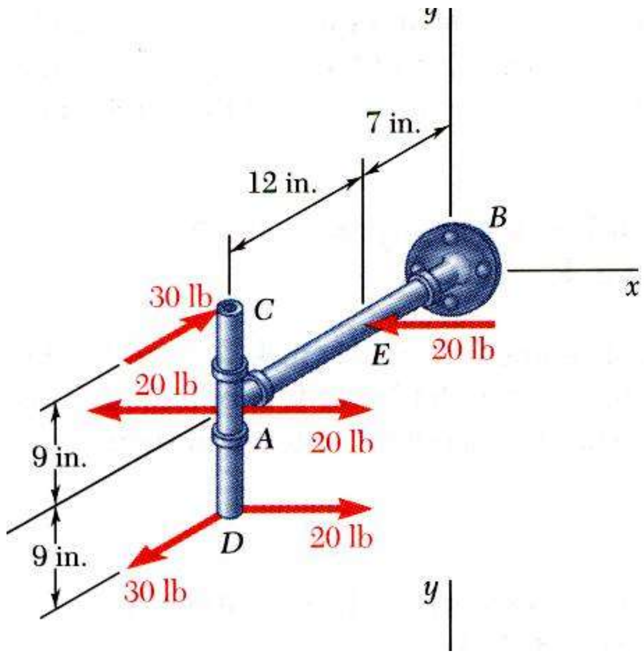
Determine the components of the single couple equivalent to the couples shown.

SOLUTION:

- Attach equal and opposite 20 lb forces in the $\pm x$ direction at A, thereby producing 3 couples for which the moment components are easily computed.

- Alternatively, compute the sum of the moments of the four forces about an arbitrary single point. The point D is a good choice as only two of the forces will produce non-zero moment contributions..

Sample Problem 3.6



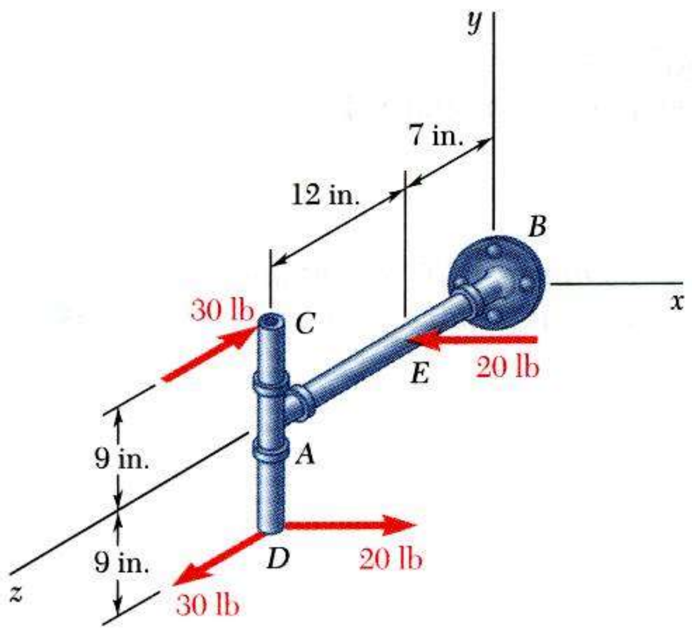
- Attach equal and opposite 20 lb forces in the $\pm x$ direction at A

- The three couples may be represented by three couple vectors,

$$M_x = -(30\,\text{lb})(18\,\text{in.}) = -540\,\text{lb}\cdot\text{in.}$$
$$M_y = +(20\,\text{lb})(12\,\text{in.}) = +240\,\text{lb}\cdot\text{in.}$$
$$M_z = +(20\,\text{lb})(9\,\text{in.}) = +180\,\text{lb}\cdot\text{in.}$$

$$\boxed{\vec{M} = -(540\,\text{lb}\cdot\text{in.})\vec{i} + (240\,\text{lb}\cdot\text{in.})\vec{j} + (180\,\text{lb}\cdot\text{in.})\vec{k}}$$

Sample Problem 3.6



- Alternatively, compute the sum of the moments of the four forces about *D*.

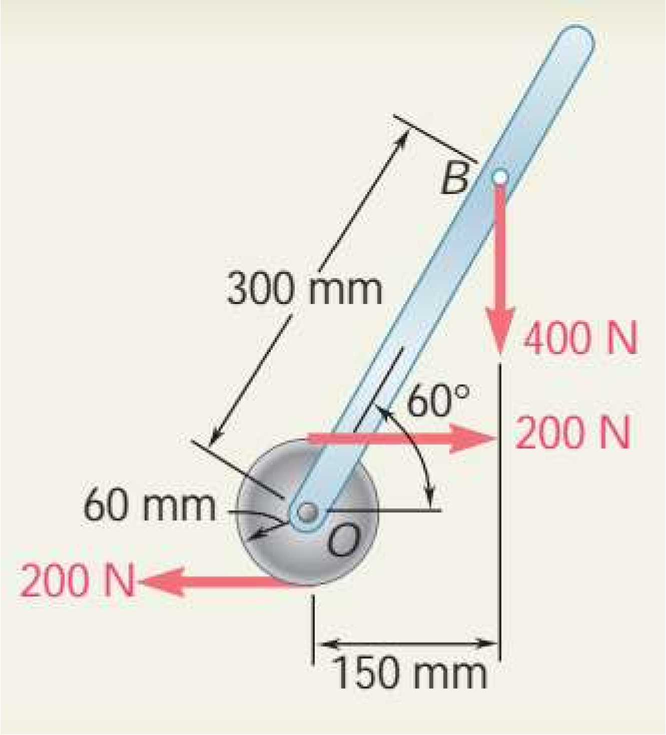
- Only the forces at *C* and *E* contribute to the moment about *D*.

$$\vec{M} = \vec{M}_D = (18\text{ in.})\vec{j} \times (-30\text{ lb})\vec{k} + \left[(9\text{ in.})\vec{j} - (12\text{ in.})\vec{k}\right] \times (-20\text{ lb})\vec{i}$$

$$\boxed{\vec{M} = -(540\text{ lb}\cdot\text{in.})\vec{i} + (240\text{lb}\cdot\text{in.})\vec{j} + (180\text{ lb}\cdot\text{in.})\vec{k}}$$

Sample Problem 3.7

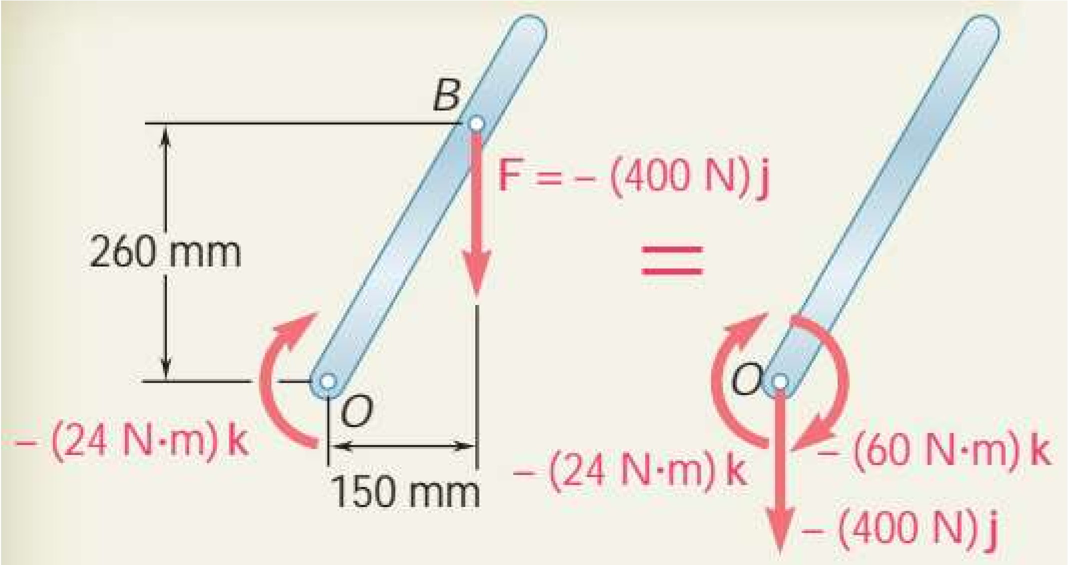
Replace the couple and force shown by an equivalent single force applied to the lever. Determine the distance from the shaft to the point of application of this equivalent force.



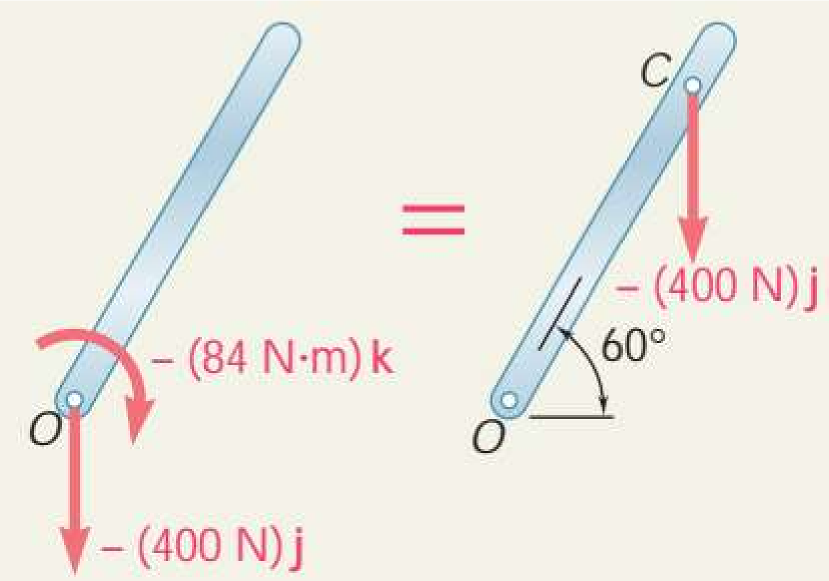
Sample Problem 3.7

First the given force and couple are replaced by an equivalent force-couple system at O. We move the force $\mathbf{F} = -(400\text{ N})\mathbf{j}$ to O and at the same time add a couple of moment $\mathbf{M}_O$ equal to the moment about O of the force in its original position.

$$\mathbf{M}_O = \overrightarrow{OB} \times \mathbf{F} = [(0.150\,\text{m})\mathbf{i} + (0.260\,\text{m})\mathbf{j}] \times (-400\,\text{N})\mathbf{j}$$
$$= -(60\text{ N}\cdot\text{m})\mathbf{k}$$



Sample Problem 3.7



Sample Problem 3.7

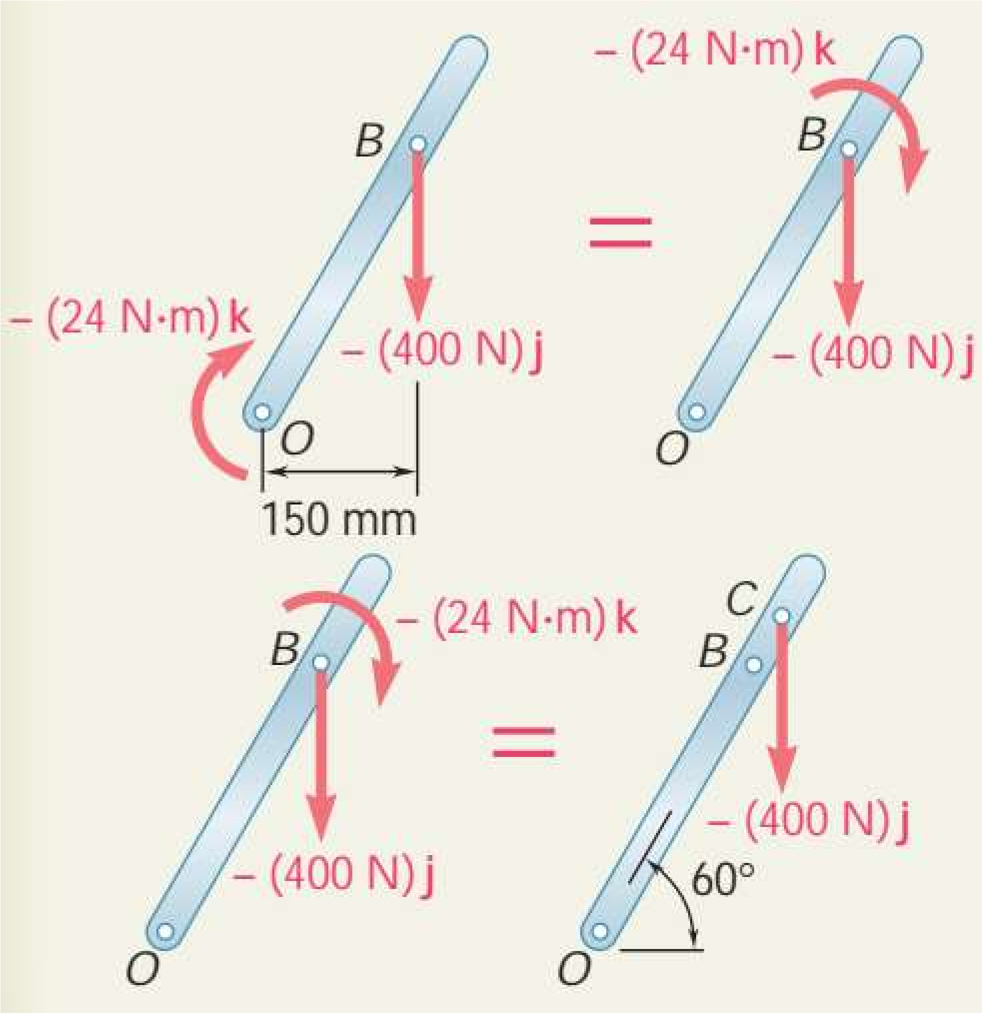
This couple is added to the couple of moment $-(24\ \text{N}\cdot\text{m})\mathbf{k}$ formed by the two 200-N forces, and a couple of moment $-(84\ \text{N}\cdot\text{m})\mathbf{k}$ is obtained. This last couple can be eliminated by applying **F** at a point C chosen in such a way that

$$
\begin{aligned}
-(84\,\text{N}\cdot\text{m})\mathbf{k} &= \overrightarrow{OC}\times\mathbf{F} \\
&= [(OC)\cos 60°\mathbf{i} + (OC)\sin 60°\mathbf{j}]\times(-400\,\text{N})\mathbf{j} \\
&= -(OC)\cos 60°(400\,\text{N})\mathbf{k}
\end{aligned}
$$

We conclude that

$$(OC)\cos 60° = 0.210\ \text{m} = 210\ \text{mm} \qquad OC = 420\ \text{mm}$$

Sample Problem 3.7



Sample Problem 3.7

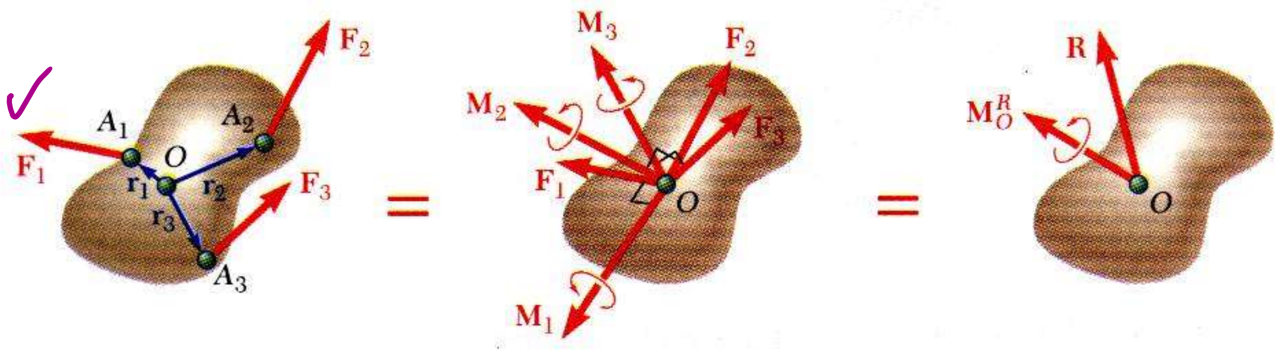
Alternative Solution. Since the effect of a couple does not depend on its location, the couple of moment $-(24 \text{ N} \cdot \text{m})\mathbf{k}$ can be moved to B; we thus obtain a force-couple system at B. The couple can now be eliminated by applying $\mathbf{F}$ at a point C chosen in such a way that

$$-(24\,\text{N}\cdot\text{m})\mathbf{k} = \overrightarrow{BC} \times \mathbf{F}$$
$$= -(BC)\cos 60°(400\,\text{N})\mathbf{k}$$

We conclude that

$$(BC)\cos 60° = 0.060 \text{ m} = 60 \text{ mm} \qquad BC = 120 \text{ mm}$$
$$OC = OB + BC = 300 \text{ mm} + 120 \text{ mm} \qquad OC = 420 \text{ mm}$$

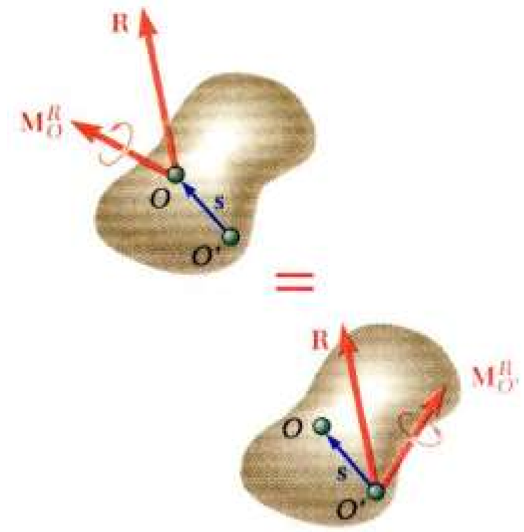
System of Forces: Reduction to a Force and Couple



- A system of forces may be replaced by a collection of force-couple systems acting a given point O

- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F} \right)$$



- The force-couple system at O may be moved to O' with the addition of the moment of $\boldsymbol{R}$ about O' ,

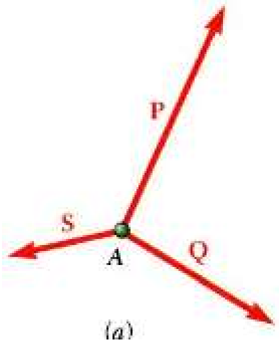
$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

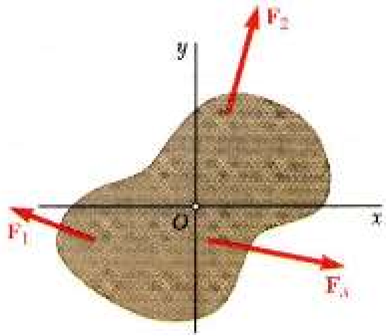
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

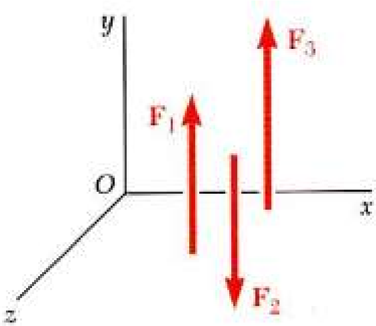
Further Reduction of a System of Forces

- If the resultant force and couple at *O* are mutually perpendicular, they can be replaced by a single force acting along a new line of action.

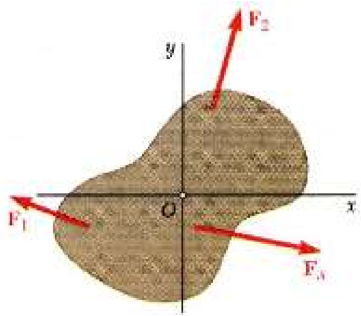
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.

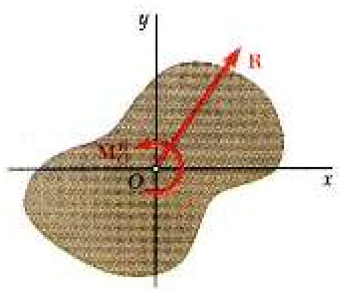


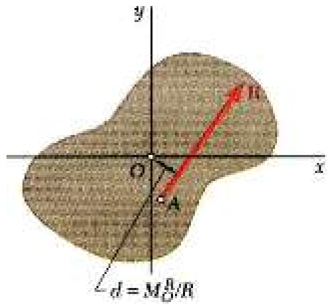




Further Reduction of a System of Forces





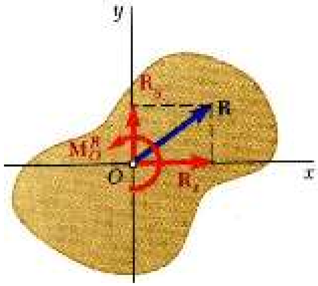


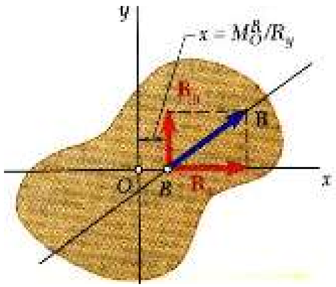
- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.

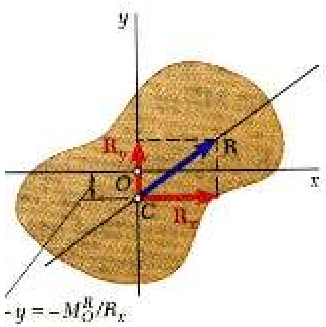
- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$

- In terms of rectangular coordinates,

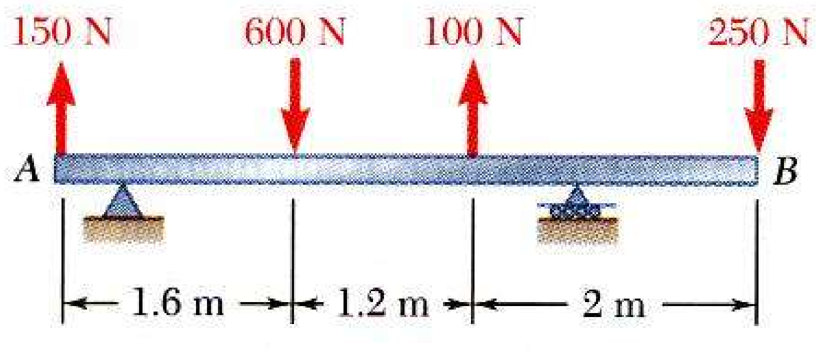
$$xR_y - yR_x = M_O^R$$







Sample Problem 3.8



For the beam, reduce the system of forces shown to (a) an equivalent force-couple system at *A*, (b) an equivalent force couple system at *B*, and (c) a single force or resultant.

Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.

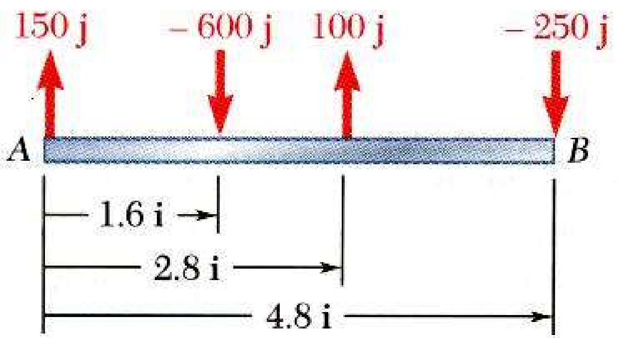
SOLUTION:

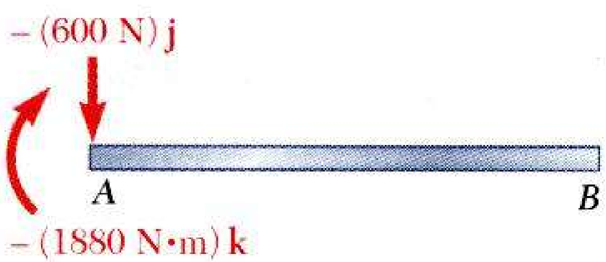
a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about *A*.

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

c) Determine the point of application for the resultant force such that its moment about *A* is equal to the resultant couple at *A*.

Sample Problem 3.8





SOLUTION:

a) Compute the resultant force and the resultant couple at A.

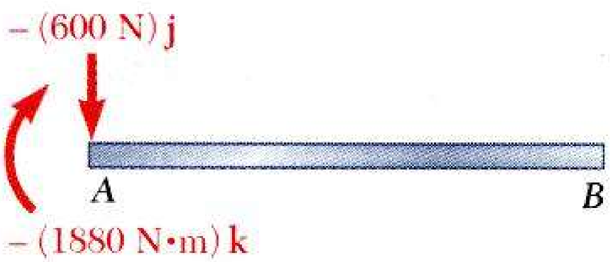
$$\vec{R} = \sum \vec{F}$$
$$= (150\text{ N})\vec{j} - (600\text{ N})\vec{j} + (100\text{ N})\vec{j} - (250\text{ N})\vec{j}$$

$$\boxed{\vec{R} = -(600\text{ N})\vec{j}}$$

$$\vec{M}_A^R = \sum\left(\vec{r} \times \vec{F}\right)$$
$$= \left(1.6\vec{i}\right)\times\left(-600\vec{j}\right) + \left(2.8\vec{i}\right)\times\left(100\vec{j}\right)$$
$$+ \left(4.8\vec{i}\right)\times\left(-250\vec{j}\right)$$

$$\boxed{\vec{M}_A^R = -(1880\text{ N}\cdot\text{m})\vec{k}}$$

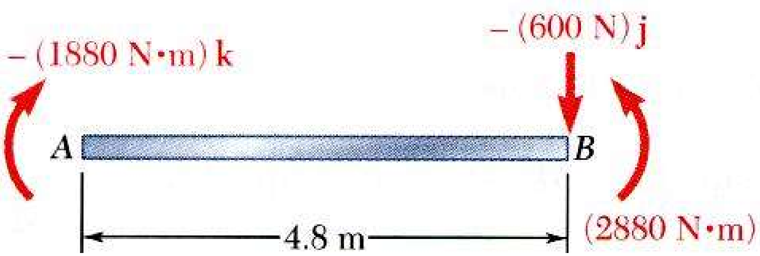
Sample Problem 3.8



b) Find an equivalent force-couple system at B based on the force-couple system at A.

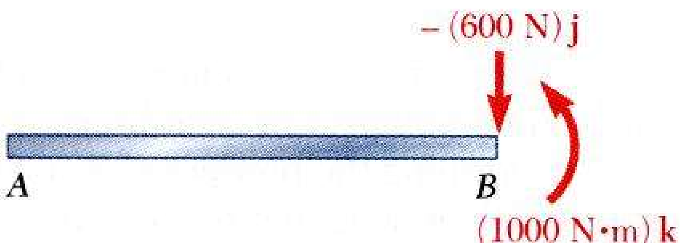
The force is unchanged by the movement of the force-couple system from A to B.

$$\boxed{\vec{R} = -(600\,\mathrm{N})\vec{j}}$$



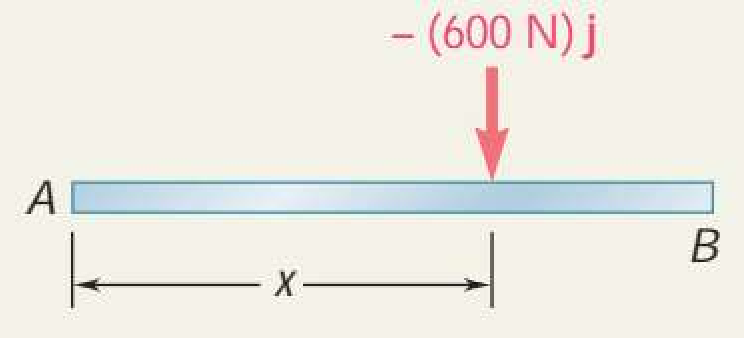
The couple at B is equal to the moment about B of the force-couple system found at A.

$$\begin{aligned}
\vec{M}_B^R &= \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R} \\
&= -(1880\,\mathrm{N\cdot m})\vec{k} + (-4.8\,\mathrm{m})\vec{i} \times (-600\,\mathrm{N})\vec{j} \\
&= -(1880\,\mathrm{N\cdot m})\vec{k} + (2880\,\mathrm{N\cdot m})\vec{k}
\end{aligned}$$



$$\boxed{\vec{M}_B^R = +(1000\,\mathrm{N\cdot m})\vec{k}}$$

Sample Problem 3.8



c. Single Force or Resultant. The resultant of the given system of forces is equal to **R**, and its point of application must be such that the moment of **R** about A is equal to $\mathbf{M}_A^R$. We write

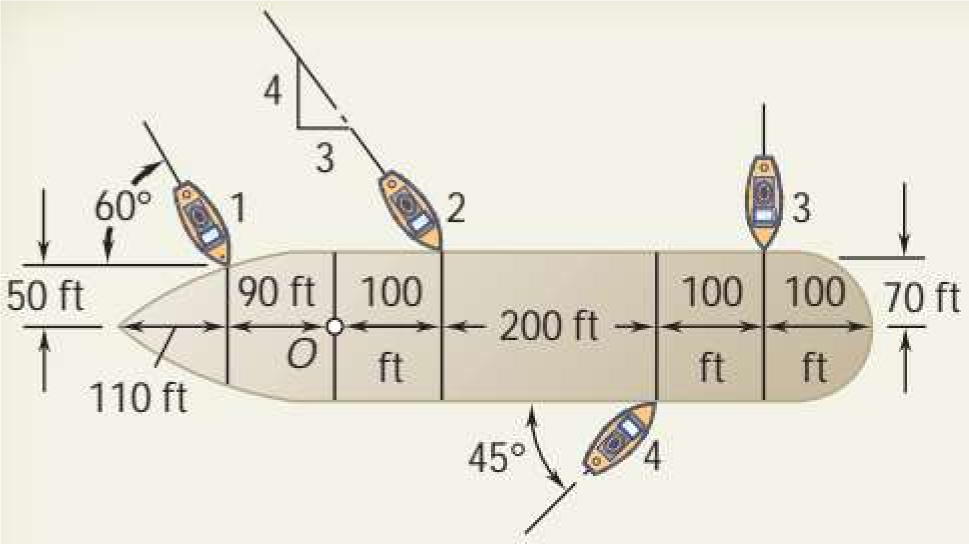
$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_A^R$$
$$x\mathbf{i} \times (-600\ \text{N})\mathbf{j} = -(1880\ \text{N} \cdot \text{m})\mathbf{k}$$
$$-x(600\ \text{N})\mathbf{k} = -(1880\ \text{N} \cdot \text{m})\mathbf{k}$$

and conclude that $x = 3.13$ m. Thus, the single force equivalent to the given system is defined as

$$\mathbf{R} = 600\ \text{N}\downarrow \qquad x = 3.13\ \text{m}$$

Sample Problem 3.9

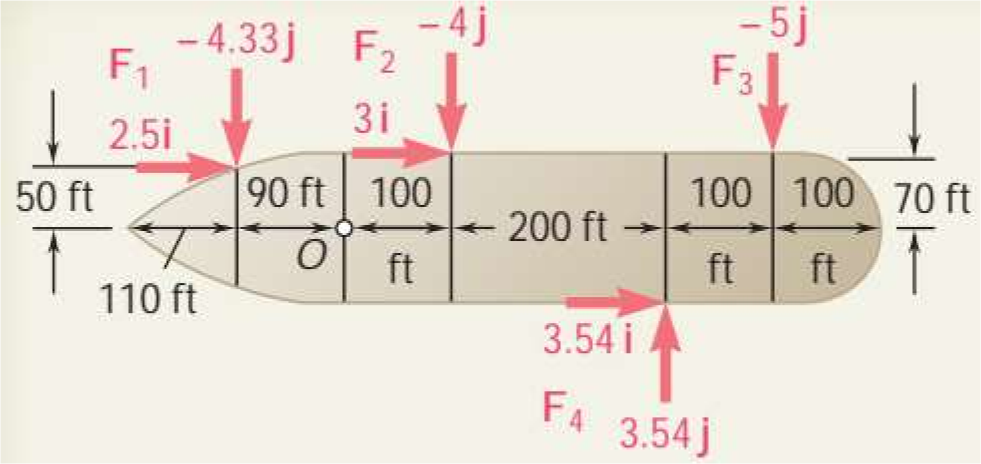
Four tugboats are used to bring an ocean liner to its pier. Each tugboat exerts a 5000-lb force in the direction shown. Determine (*a*) the equivalent force-couple system at the foremast *O*, (*b*) the point on the hull where a single, more powerful tugboat should push to produce the same effect as the original four tugboats.



Sample Problem 3.9

a. Force-Couple System at O. Each of the given forces is resolved into components in the diagram shown (kip units are used). The force-couple system at O equivalent to the given system of forces consists of a force $\mathbf{R}$ and a couple $\mathbf{M}_O^R$ defined as follows:

$$\begin{aligned}\mathbf{R} &= \Sigma\mathbf{F}\\ &= (2.50\mathbf{i} - 4.33\mathbf{j}) + (3.00\mathbf{i} - 4.00\mathbf{j}) + (-5.00\mathbf{j}) + (3.54\mathbf{i} + 3.54\mathbf{j})\\ &= 9.04\mathbf{i} - 9.79\mathbf{j}\end{aligned}$$



Sample Problem 3.9

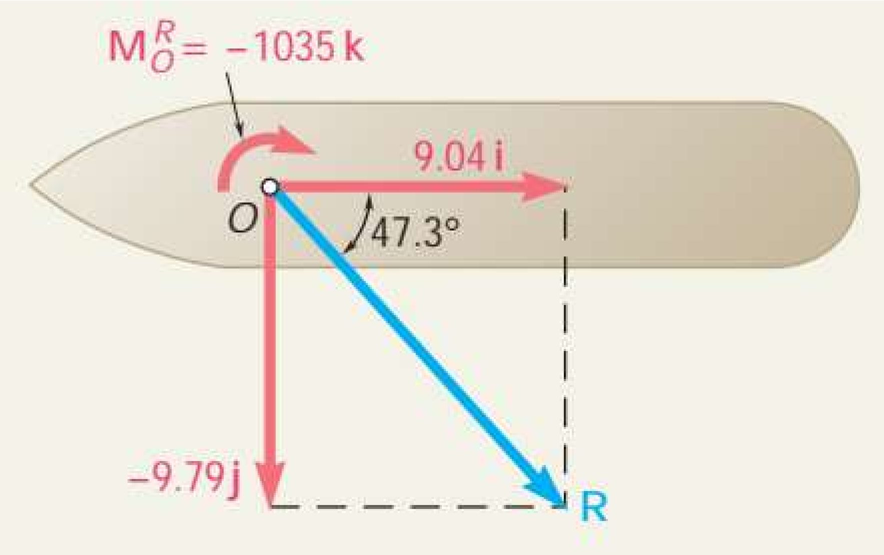
$$
\begin{aligned}
\mathbf{M}_O^R &= \Sigma(\mathbf{r} \times \mathbf{F}) \\
&= (-90\mathbf{i} + 50\mathbf{j}) \times (2.50\mathbf{i} - 4.33\mathbf{j}) \\
&\quad + (100\mathbf{i} + 70\mathbf{j}) \times (3.00\mathbf{i} - 4.00\mathbf{j}) \\
&\quad + (400\mathbf{i} + 70\mathbf{j}) \times (-5.00\mathbf{j}) \\
&\quad + (300\mathbf{i} - 70\mathbf{j}) \times (3.54\mathbf{i} + 3.54\mathbf{j}) \\
&= (390 - 125 - 400 - 210 - 2000 + 1062 + 248)\mathbf{k} \\
&= -1035\mathbf{k}
\end{aligned}
$$

The equivalent force-couple system at O is thus

$$\mathbf{R} = (9.04 \text{ kips})\mathbf{i} - (9.79 \text{ kips})\mathbf{j} \qquad \mathbf{M}_O^R = -(1035 \text{ kip} \cdot \text{ft})\mathbf{k}$$

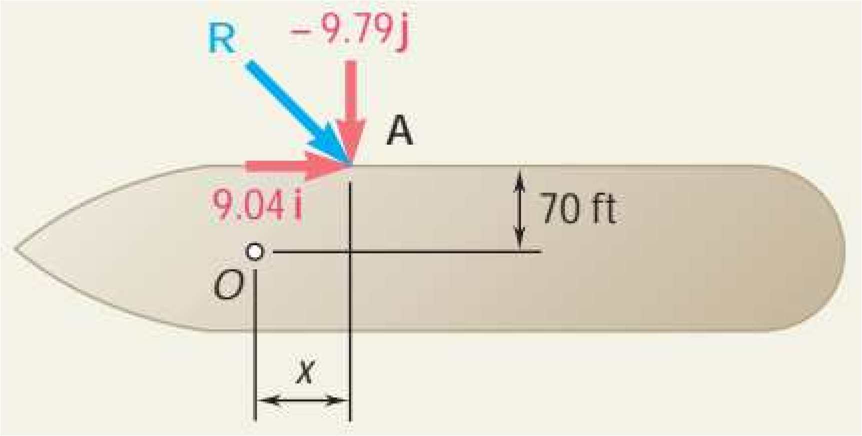
or $\qquad \mathbf{R} = 13.33 \text{ kips} \text{ c } 47.3° \qquad \mathbf{M}_O^R = 1035 \text{ kip} \cdot \text{ft i}$

Sample Problem 3.9



Remark. Since all the forces are contained in the plane of the figure, we could have expected the sum of their moments to be perpendicular to that plane. Note that the moment of each force component could have been obtained directly from the diagram by first forming the product of its magnitude and perpendicular distance to *O* and then assigning to this product a positive or a negative sign depending upon the sense of the moment.

Sample Problem 3.9



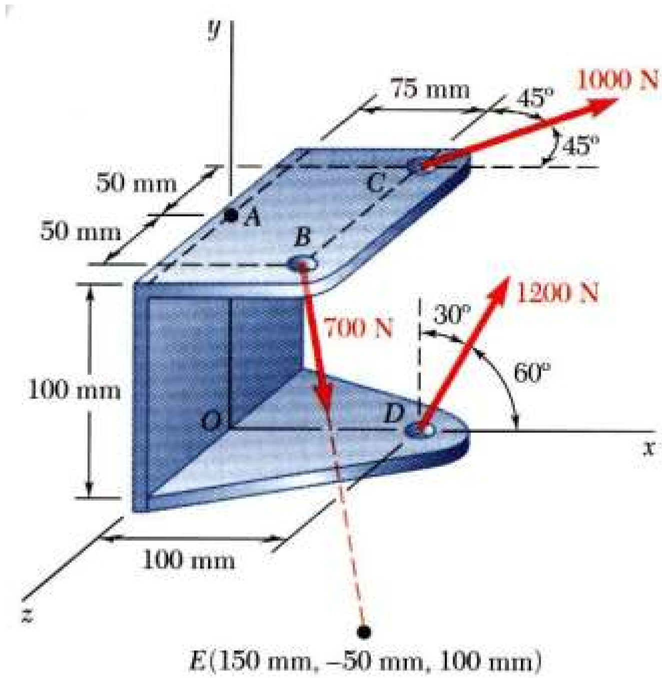
b. Single Tugboat. The force exerted by a single tugboat must be equal to **R**, and its point of application *A* must be such that the moment of **R** about *O* is equal to $\mathbf{M}_O^R$. Observing that the position vector of *A* is

$$\mathbf{r} = x\mathbf{i} + 70\mathbf{j}$$

we write

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$
$$(x\mathbf{i} + 70\mathbf{j}) \times (9.04\mathbf{i} - 9.79\mathbf{j}) = -1035\mathbf{k}$$
$$-x(9.79)\mathbf{k} - 633\mathbf{k} = -1035\mathbf{k} \qquad x = 41.1 \text{ ft}$$

Sample Problem 3.10



Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at A.

SOLUTION:

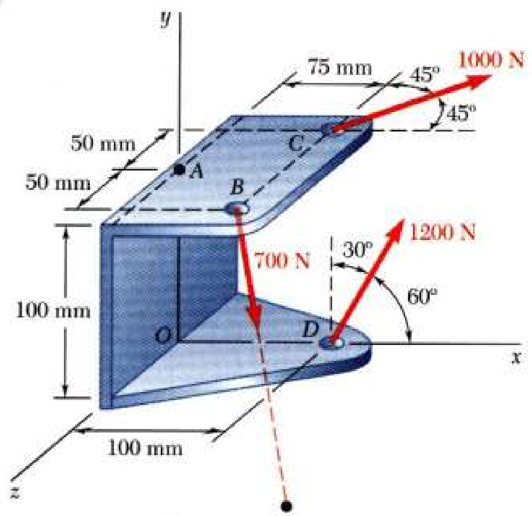
- Determine the relative position vectors for the points of application of the cable forces with respect to A.
- Resolve the forces into rectangular components.
- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$

- Compute the equivalent couple,

$$\vec{M}_A^R = \sum\left(\vec{r} \times \vec{F}\right)$$

Sample Problem 3.10



<u>SOLUTION:</u>

- Determine the relative position vectors with respect to *A*.

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k} \ (\mathrm{m})$$

$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k} \ (\mathrm{m})$$

$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j} \ (\mathrm{m})$$

- Resolve the forces into rectangular components.

$$\vec{F}_B = (700\ \mathrm{N})\vec{\lambda}$$

$$\vec{\lambda} = \frac{\vec{r}_{E/B}}{r_{E/B}} = \frac{75\vec{i} - 150\vec{j} + 50\vec{k}}{175}$$

$$= 0.429\vec{i} - 0.857\vec{j} + 0.289\vec{k}$$

$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k} \ (\mathrm{N})$$

$$\vec{F}_C = (1000\ \mathrm{N})\left(\cos 45\vec{i} - \cos 45\vec{k}\right)$$

$$= 707\vec{i} - 707\vec{k} \ (\mathrm{N})$$

$$\vec{F}_D = (1200\ \mathrm{N})\left(\cos 60\vec{i} + \cos 30\vec{j}\right)$$

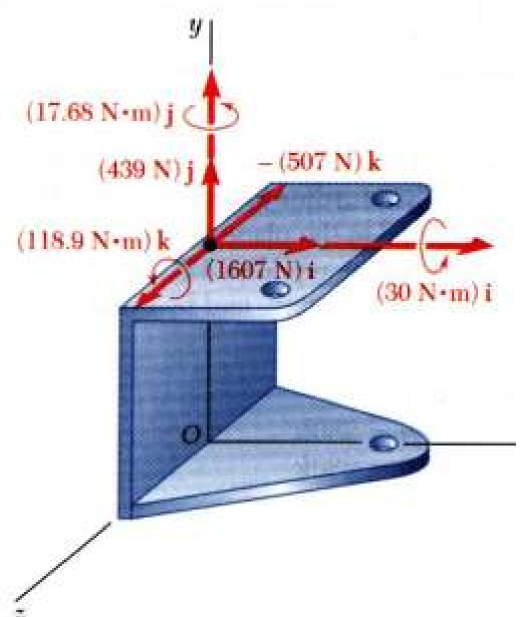
$$= 600\vec{i} + 1039\vec{j} \ (\mathrm{N})$$

Sample Problem 3.10

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i}$$
$$+ (-600 + 1039)\vec{j}$$
$$+ (200 - 707)\vec{k}$$

$$\boxed{\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k} \ \ (\text{N})}$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum(\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

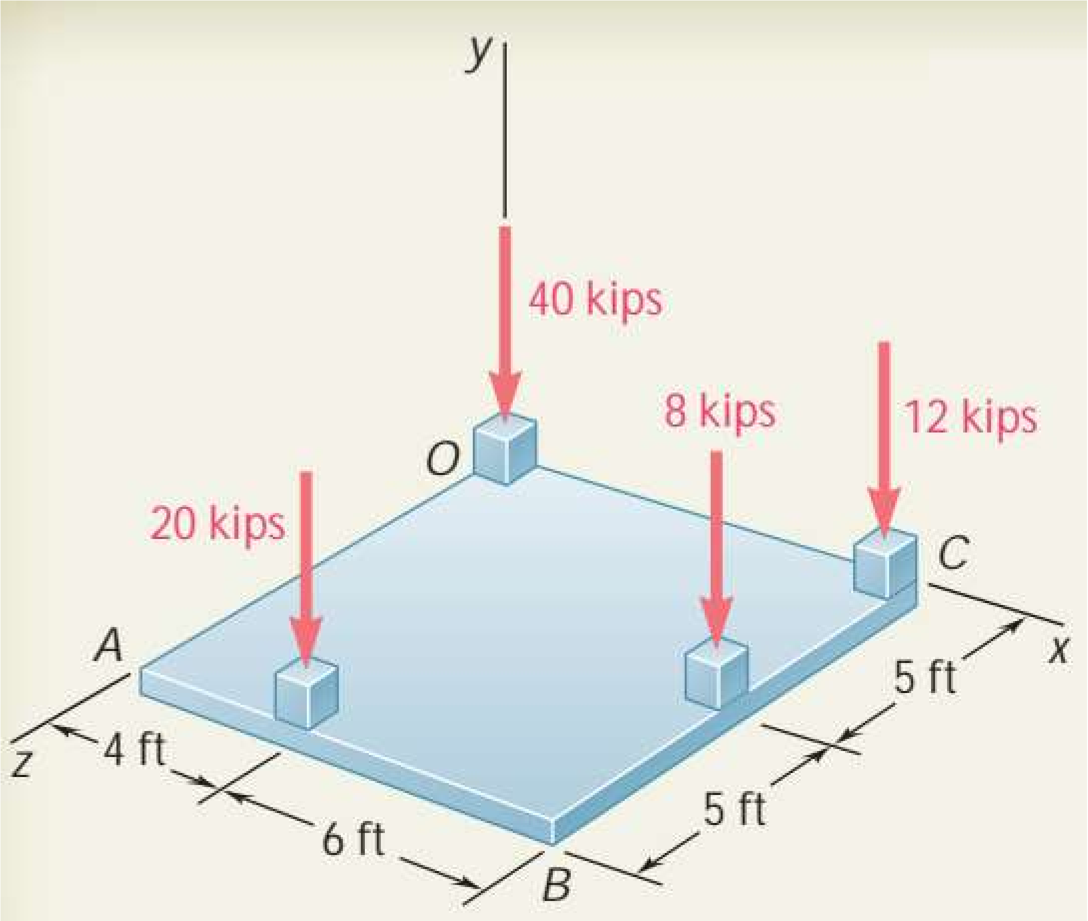
$$\vec{r}_{C/A} \times \vec{F}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\boxed{\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}}$$

Sample Problem 3.11

A square foundation mat supports the four columns shown. Determine the magnitude and point of application of the resultant of the four loads.



Sample Problem 3.11

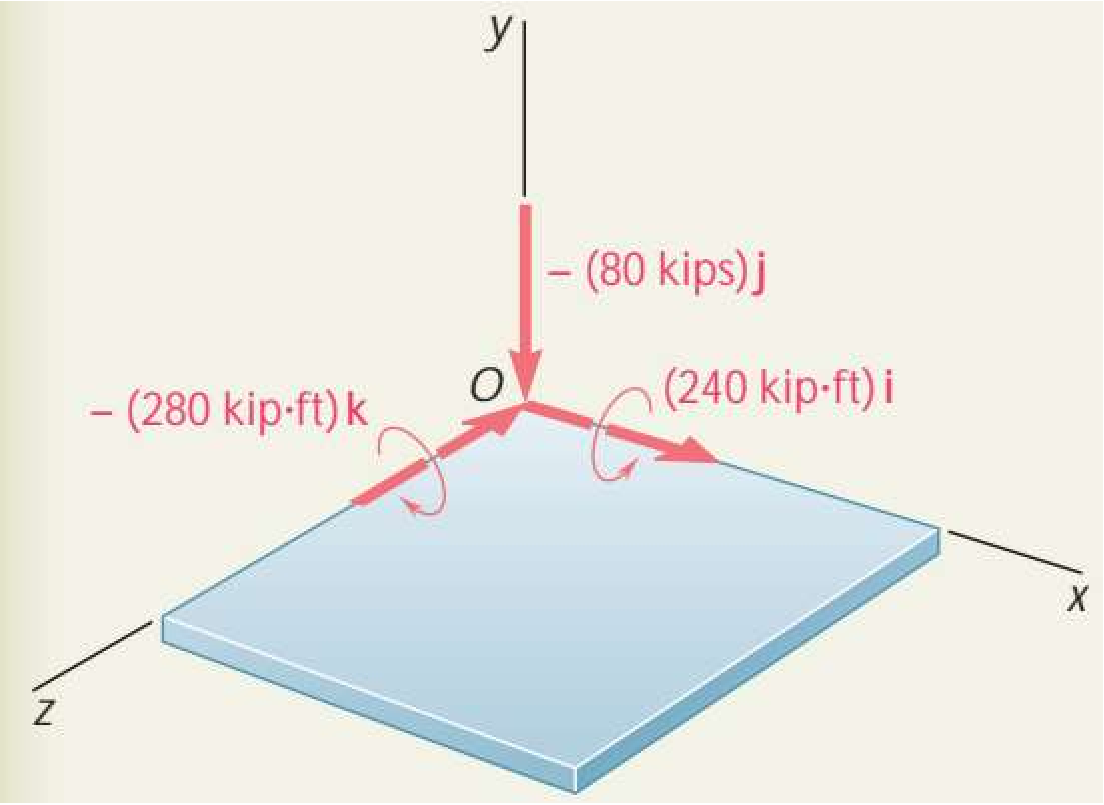
We first reduce the given system of forces to a force-couple system at the origin O of the coordinate system. This force-couple system consists of a force $\mathbf{R}$ and a couple vector $\mathbf{M}_O^R$ defined as follows:

$$\mathbf{R} = \Sigma \mathbf{F} \qquad \mathbf{M}_O^R = \Sigma(\mathbf{r} \times \mathbf{F})$$

The position vectors of the points of application of the various forces are determined, and the computations are arranged in tabular form.

$\mathbf{r}$, ft	$\mathbf{F}$, kips	$\mathbf{r} \times \mathbf{F}$, kip · ft
0	$-40\mathbf{j}$	0
$10\mathbf{i}$	$-12\mathbf{j}$	$-120\mathbf{k}$
$10\mathbf{i} + 5\mathbf{k}$	$-8\mathbf{j}$	$40\mathbf{i} - 80\mathbf{k}$
$4\mathbf{i} + 10\mathbf{k}$	$-20\mathbf{j}$	$200\mathbf{i} - 80\mathbf{k}$
	$\mathbf{R} = -80\mathbf{j}$	$\mathbf{M}_O^R = 240\mathbf{i} - 280\mathbf{k}$

Sample Problem 3.11



Sample Problem 3.11

Since the force **R** and the couple vector $\mathbf{M}_O^R$ are mutually perpendicular, the force-couple system obtained can be reduced further to a single force **R**. The new point of application of **R** will be selected in the plane of the mat and in such a way that the moment of **R** about O will be equal to $\mathbf{M}_O^R$. Denoting by **r** the position vector of the desired point of application, and by x and z its coordinates, we write

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O^R$$
$$(x\mathbf{i} + z\mathbf{k}) \times (-80\mathbf{j}) = 240\mathbf{i} - 280\mathbf{k}$$
$$-80x\mathbf{k} + 80z\mathbf{i} = 240\mathbf{i} - 280\mathbf{k}$$

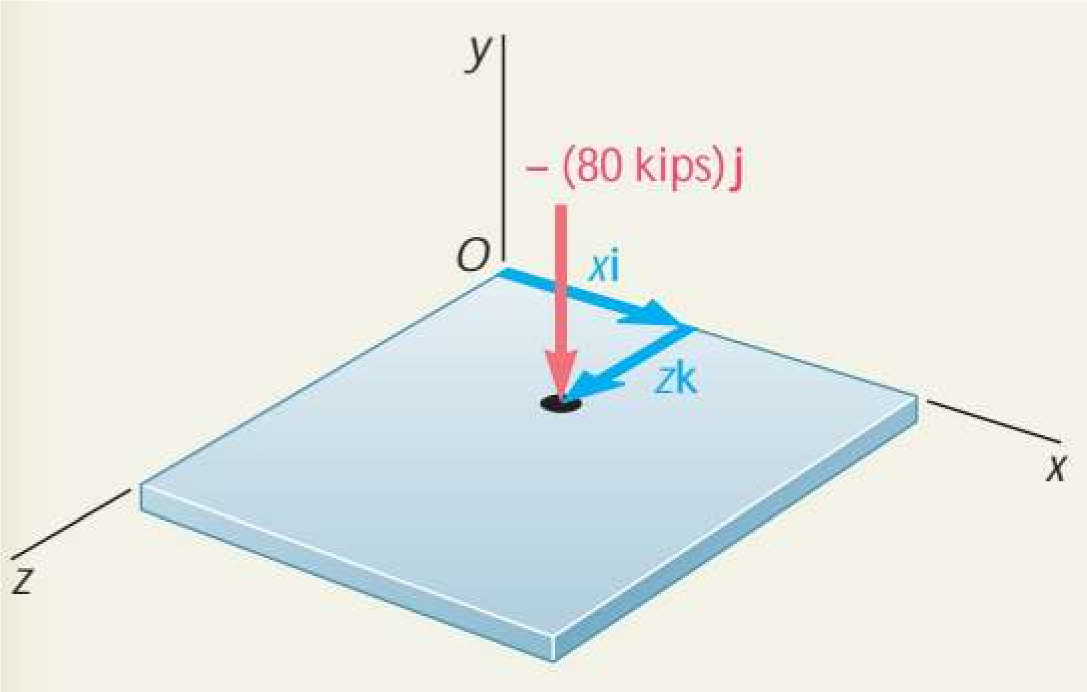
from which it follows that

$$-80x = -280 \qquad 80z = 240$$
$$x = 3.50 \text{ ft} \qquad z = 3.00 \text{ ft}$$

We conclude that the resultant of the given system of forces is

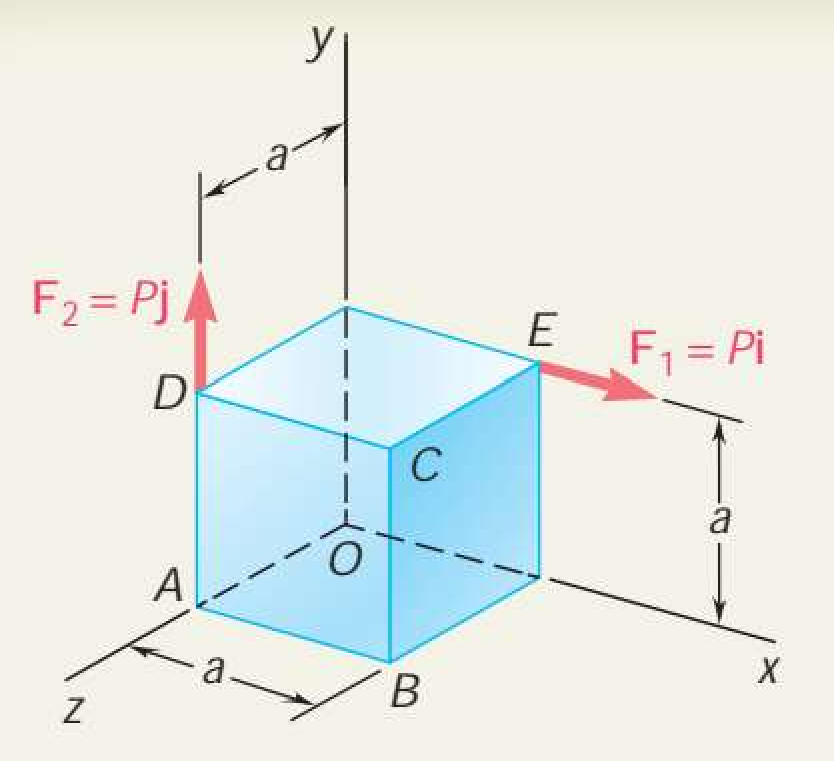
$$\mathbf{R} = 80 \text{ kips}\downarrow \qquad \text{at } x = 3.50 \text{ ft}, z = 3.00 \text{ ft}$$

Sample Problem 3.11



Sample Problem 3.12

Two forces of the same magnitude P act on a cube of side a as shown. Replace the two forces by an equivalent wrench, and determine (*a*) the magnitude and direction of the resultant force $\mathbf{R}$, (*b*) the pitch of the wrench, (*c*) the point where the axis of the wrench intersects the yz plane.



Sample Problem 3.12

Equivalent Force-Couple System at O. We first determine the equivalent force-couple system at the origin O. We observe that the position vectors of the points of application E and D of the two given forces are $\mathbf{r}_E = a\mathbf{i} + a\mathbf{j}$ and $\mathbf{r}_D = a\mathbf{j} + a\mathbf{k}$. The resultant $\mathbf{R}$ of the two forces and their moment resultant $\mathbf{M}_O^R$ about O are

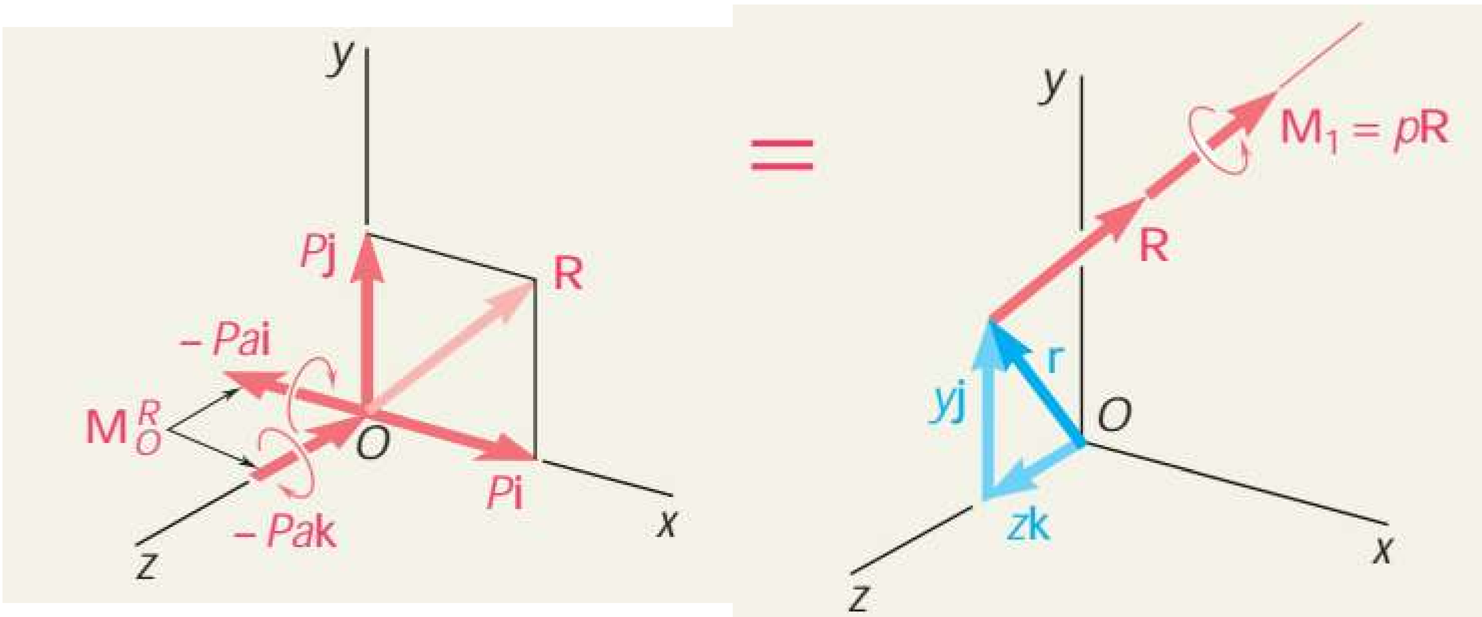
$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = P\mathbf{i} + P\mathbf{j} = P(\mathbf{i} + \mathbf{j}) \qquad (1)$$

$$\begin{aligned} \mathbf{M}_O^R &= \mathbf{r}_E \times \mathbf{F}_1 + \mathbf{r}_D \times \mathbf{F}_2 = (a\mathbf{i} + a\mathbf{j}) \times P\mathbf{i} + (a\mathbf{j} + a\mathbf{k}) \times P\mathbf{j} \\ &= -Pa\mathbf{k} - Pa\mathbf{i} = -Pa(\mathbf{i} + \mathbf{k}) \qquad (2) \end{aligned}$$

a. Resultant Force R. It follows from Eq. (1) and the adjoining sketch that the resultant force $\mathbf{R}$ has the magnitude $R = P\sqrt{2}$, lies in the xy plane, and forms angles of 45° with the x and y axes. Thus

$$R = P\sqrt{2} \qquad \theta_x = \theta_y = 45° \qquad \theta_z = 90° \quad ◀$$

Sample Problem 3.12



Sample Problem 3.12

b. Pitch of Wrench. Recalling formula (3.62) of Sec. 3.21 and Eqs. (1) and (2) above, we write

$$p = \frac{\mathbf{R}\cdot\mathbf{M}_O^R}{R^2} = \frac{P(\mathbf{i}+\mathbf{j})\cdot(-Pa)(\mathbf{i}+\mathbf{k})}{(P\sqrt{2})^2} = \frac{-P^2a(1+0+0)}{2P^2} \qquad p = -\frac{a}{2}$$

c. Axis of Wrench. It follows from the above and from Eq. (3.61) that the wrench consists of the force $\mathbf{R}$ found in (1) and the couple vector

$$\mathbf{M}_1 = p\mathbf{R} = -\frac{a}{2}P(\mathbf{i}+\mathbf{j}) = -\frac{Pa}{2}(\mathbf{i}+\mathbf{j}) \tag{3}$$

To find the point where the axis of the wrench intersects the yz plane, we express that the moment of the wrench about O is equal to the moment resultant $\mathbf{M}_O^R$ of the original system:

$$\mathbf{M}_1 + \mathbf{r}\times\mathbf{R} = \mathbf{M}_O^R$$

Sample Problem 3.12

or, noting that $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$ and substituting for $\mathbf{R}$, $\mathbf{M}_O^R$, and $\mathbf{M}_1$ from Eqs. (1), (2), and (3),

$$-\frac{Pa}{2}(\mathbf{i} + \mathbf{j}) + (y\mathbf{j} + z\mathbf{k}) \times P(\mathbf{i} + \mathbf{j}) = -Pa(\mathbf{i} + \mathbf{k})$$

$$-\frac{Pa}{2}\mathbf{i} - \frac{Pa}{2}\mathbf{j} - Py\mathbf{k} + Pz\mathbf{j} - Pz\mathbf{i} = -Pa\mathbf{i} - Pa\mathbf{k}$$

Equating the coefficients of $\mathbf{k}$, and then the coefficients of $\mathbf{j}$, we find

$$y = a \qquad z = a/2$$

CHAPTER

4

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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Equilibrium of Rigid Bodies

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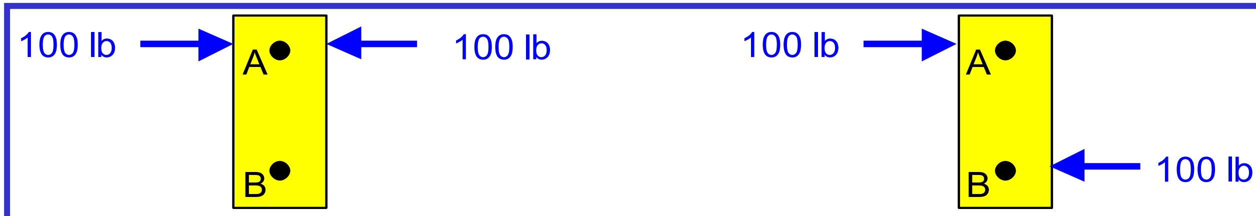
Contents

Introduction

Chapter 4 – Systems of Forces and Moments

Ch. 2: Forces were applied at a single point, so equilibrium would occur if $\Sigma F = 0$

Ch. 4: Forces are applied at various points. In some cases, these forces will give the object a tendency to rotate and $\Sigma F = 0$ is not enough to insure equilibrium.



Case 1: The object is in equilibrium
$\Sigma F = 0$

Case 2: The object is NOT in equilibrium
$\Sigma F = 0$

In Case 2 above, the object has a tendency to rotate (a *moment*, M), so the object is not in equilibrium. Equilibrium for Case 2 also requires that the sum of the moments equal zero, indicating that there is no tendency to rotate.

So the *general conditions for equilibrium* are:

$$\boxed{\begin{array}{ll} \Sigma F = 0 & \text{(no translation)} \\ \Sigma M = 0 & \text{(no rotation)} \end{array}}$$

Introduction

- For a rigid body in static equilibrium, the external forces and moments are balanced and will impart no translational or rotational motion to the body.

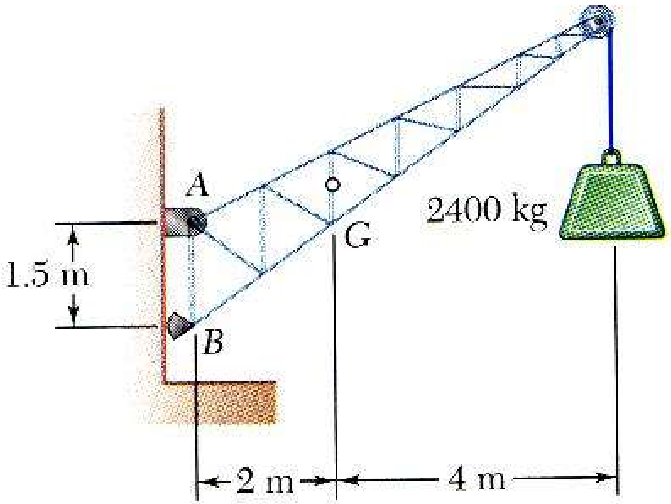
- The necessary and sufficient condition for the static equilibrium of a body are that the resultant force and couple from all external forces form a system equivalent to zero,

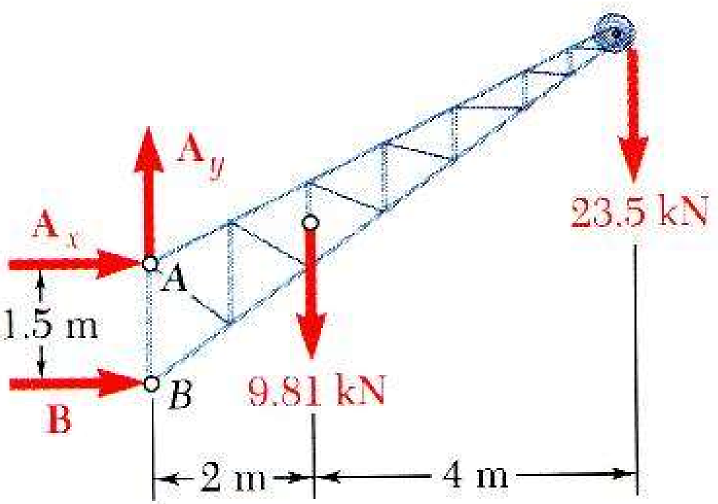
$$\sum \vec{F} = 0 \quad \sum \vec{M}_O = \sum\left(\vec{r} \times \vec{F}\right) = 0$$

- Resolving each force and moment into its rectangular components leads to 6 scalar equations which also express the conditions for static equilibrium,

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

Free-Body Diagram

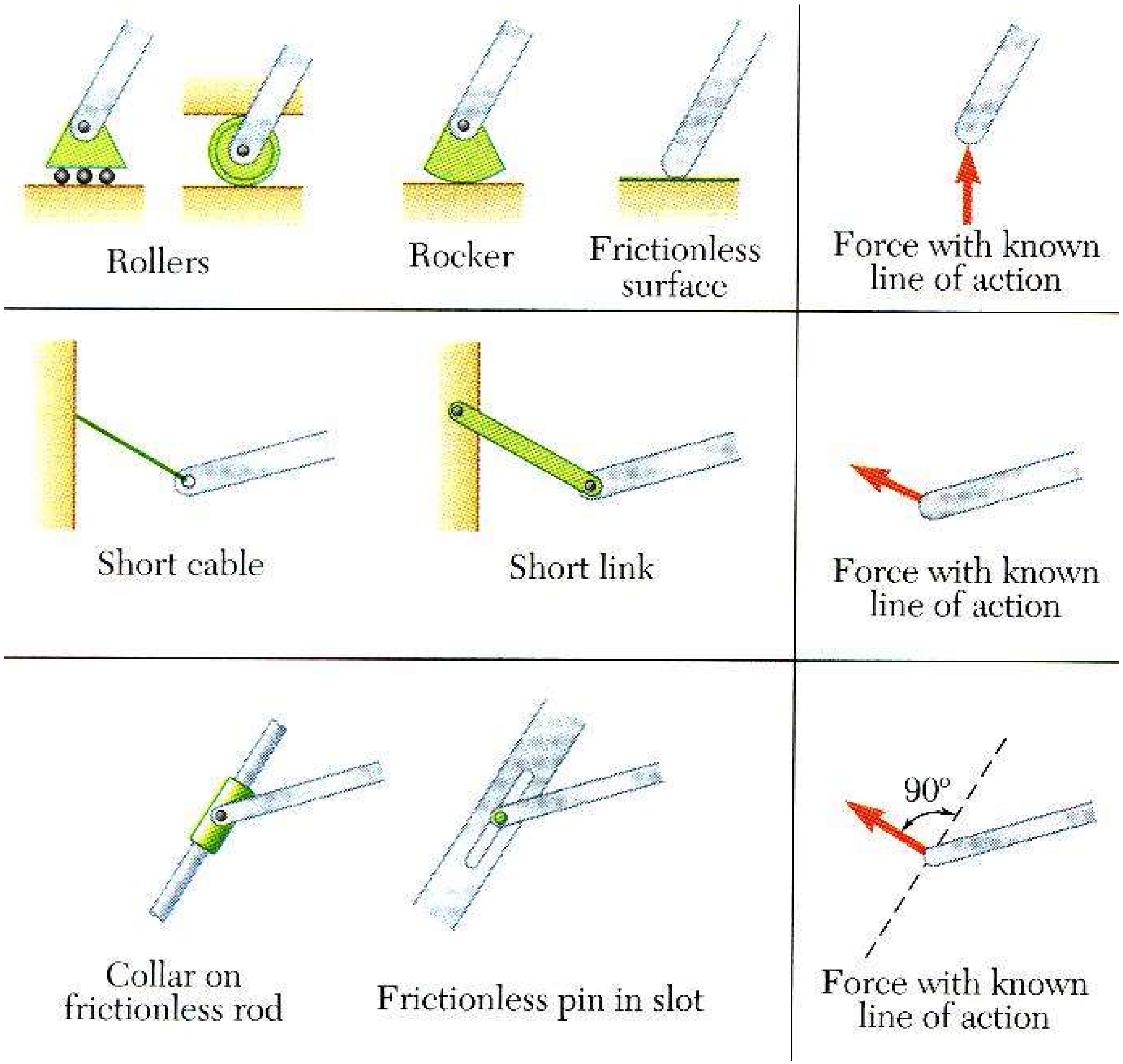




First step in the static equilibrium analysis of a rigid body is identification of all forces acting on the body with a *free-body* diagram.

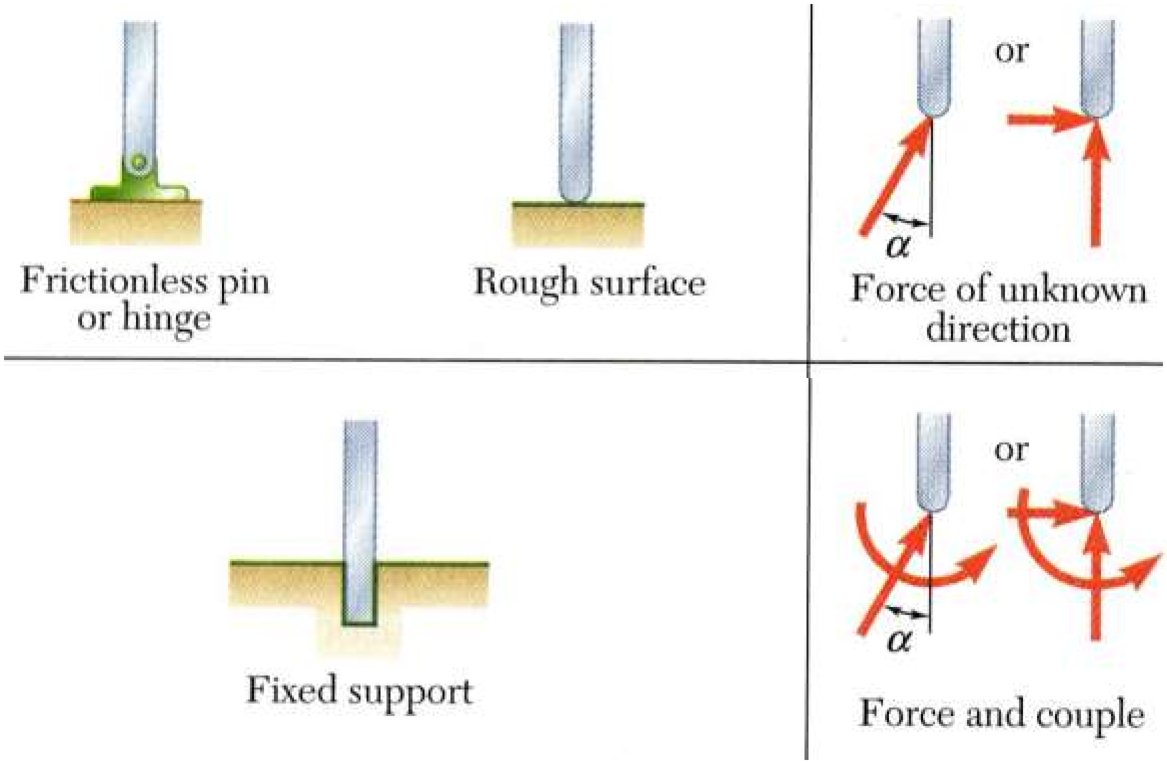
- Select the extent of the free-body and detach it from the ground and all other bodies.
- Indicate point of application, magnitude, and direction of external forces, including the rigid body weight.
- Indicate point of application and assumed direction of unknown applied forces. These usually consist of reactions through which the ground and other bodies oppose the possible motion of the rigid body.
- Include the dimensions necessary to compute the moments of the forces.

Reactions at Supports and Connections for a Two-Dimensional Structure



- Reactions equivalent to a force with known line of action.

Reactions at Supports and Connections for a Two-Dimensional Structure

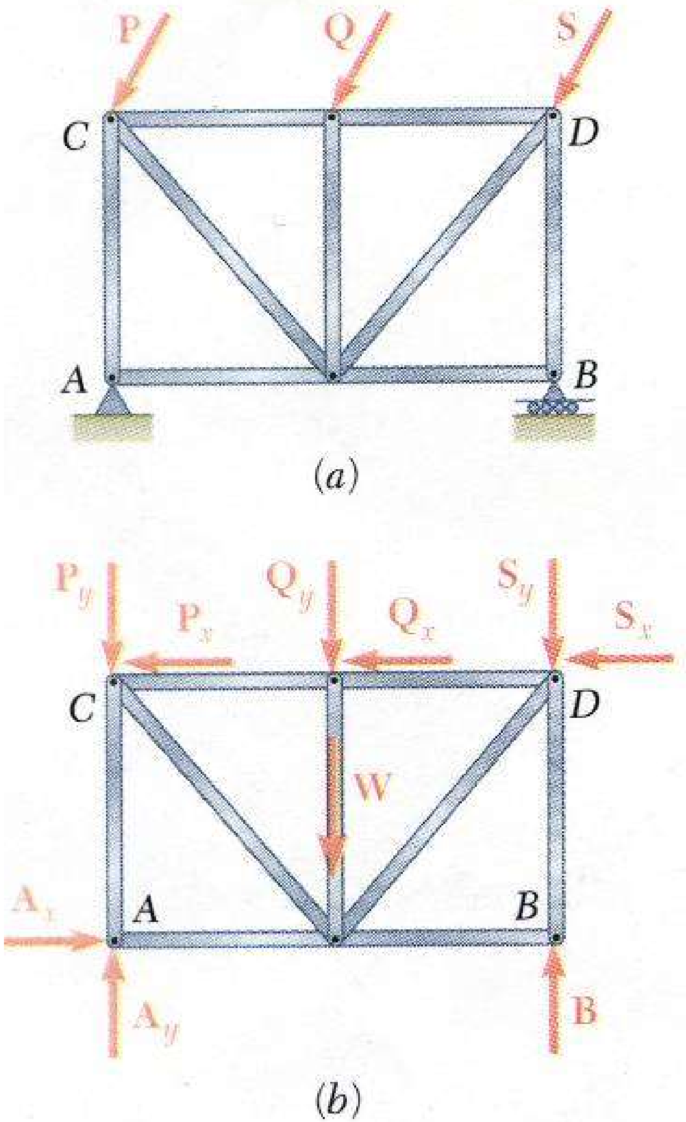


- Reactions equivalent to a force of unknown direction and magnitude.

- Reactions equivalent to a force of unknown direction and magnitude and a couple.of unknown magnitude

Examples of supports

Equilibrium of a Rigid Body in Two Dimensions



- For all forces and moments acting on a two-dimensional structure,

$$F_z = 0 \quad M_x = M_y = 0 \quad M_z = M_O$$

- Equations of equilibrium become

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

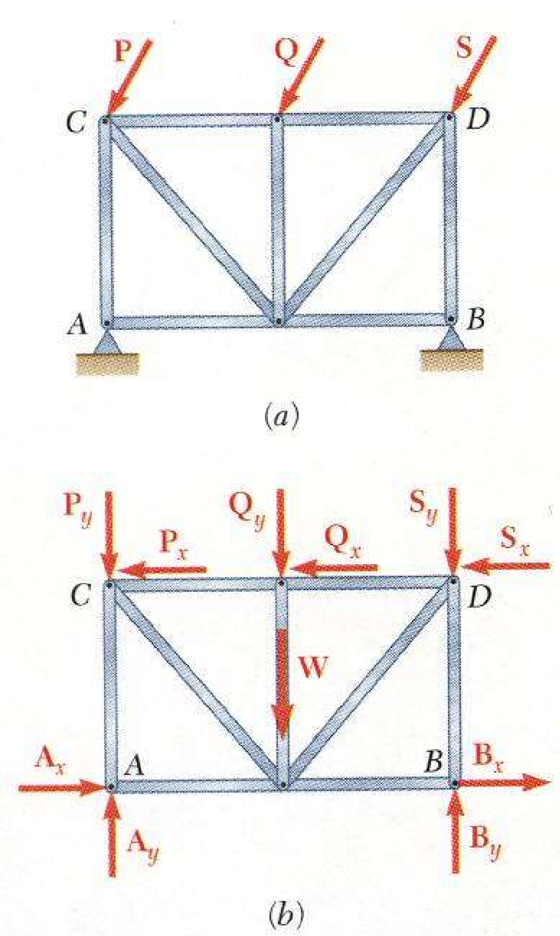
where A is any point in the plane of the structure.

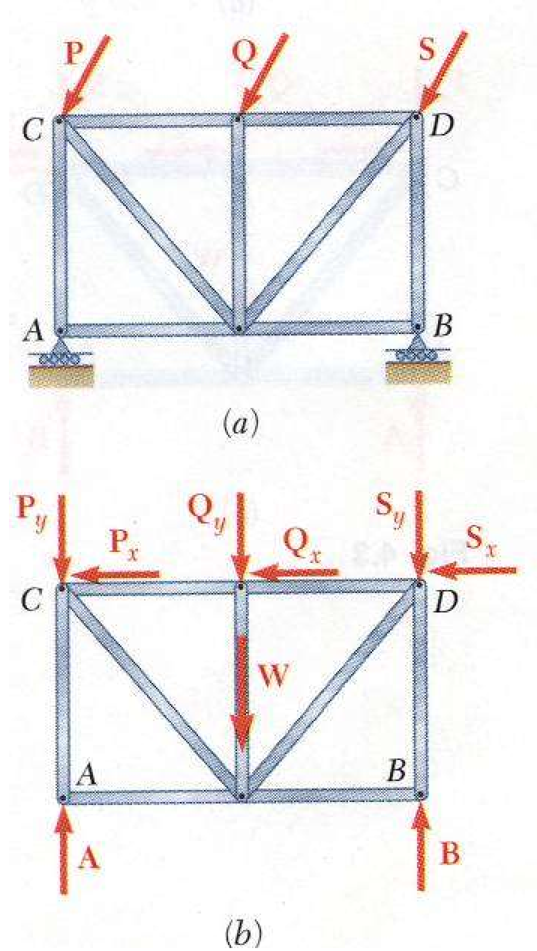
- The 3 equations can be solved for no more than 3 unknowns.

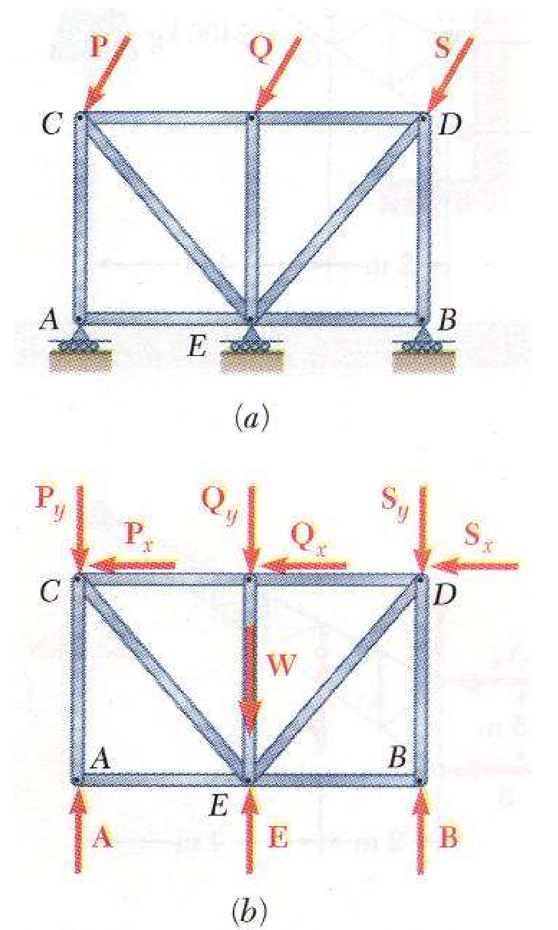
- The 3 equations can not be augmented with additional equations, but they can be replaced

$$\sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0$$

Statically Indeterminate Reactions

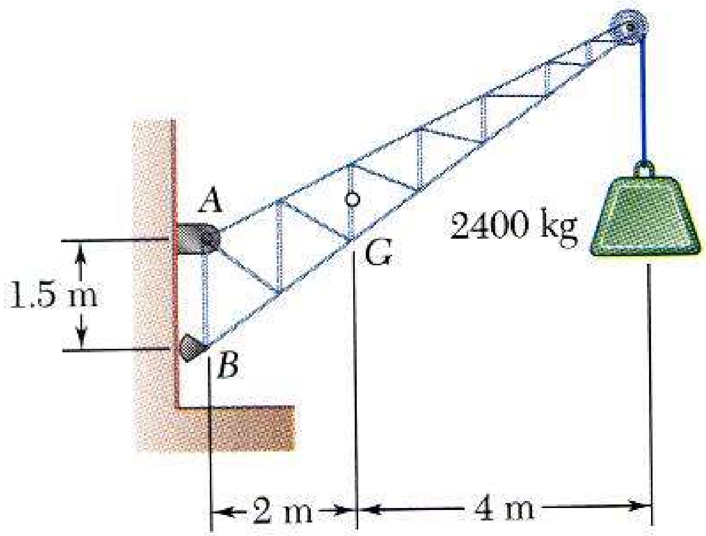






- More unknowns than equations
- Fewer unknowns than equations, partially constrained
- Equal number unknowns and equations but improperly constrained

Sample Problem 4.1



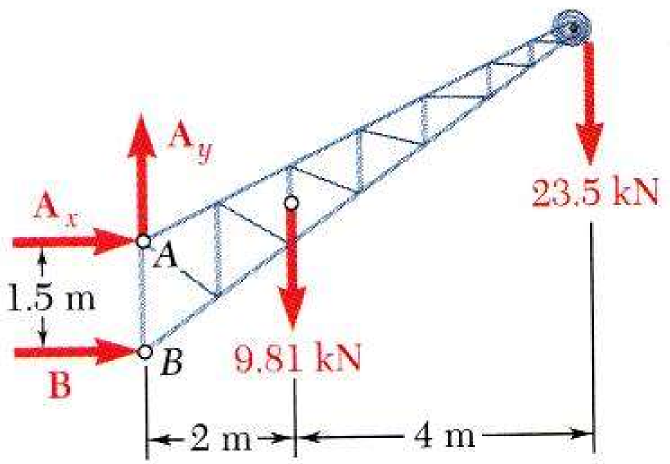
A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held in place by a pin at *A* and a rocker at *B*. The center of gravity of the crane is located at *G*.

Determine the components of the reactions at *A* and *B*.

SOLUTION:

- Create a free-body diagram for the crane.
- Determine *B* by solving the equation for the sum of the moments of all forces about *A*. Note there will be no contribution from the unknown reactions at *A*.
- Determine the reactions at *A* by solving the equations for the sum of all horizontal force components and all vertical force components.
- Check the values obtained for the reactions by verifying that the sum of the moments about *B* of all forces is zero.

Sample Problem 4.1



- Create the free-body diagram.

- Determine B by solving the equation for the sum of the moments of all forces about A.

$$\sum M_A = 0: \quad +B(1.5\text{m}) - 9.81\,\text{kN}(2\text{m}) - 23.5\,\text{kN}(6\text{m}) = 0$$

$$\boxed{B = +107.1\,\text{kN}}$$

- Determine the reactions at A by solving the equations for the sum of all horizontal forces and all vertical forces.

$$\sum F_x = 0: \quad A_x + B = 0$$

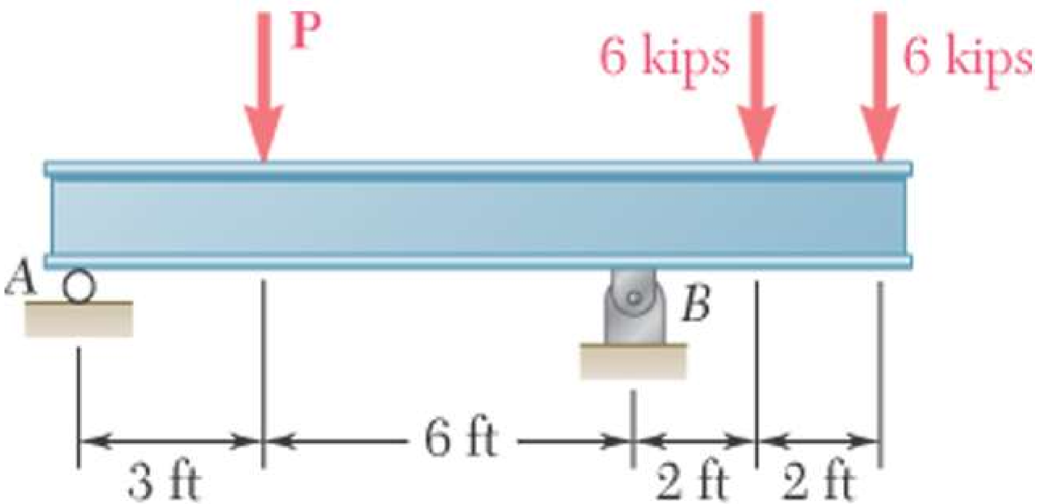
$$\boxed{A_x = -107.1\,\text{kN}}$$

$$\sum F_y = 0: \quad A_y - 9.81\,\text{kN} - 23.5\,\text{kN} = 0$$

$$\boxed{A_y = +33.3\,\text{kN}}$$

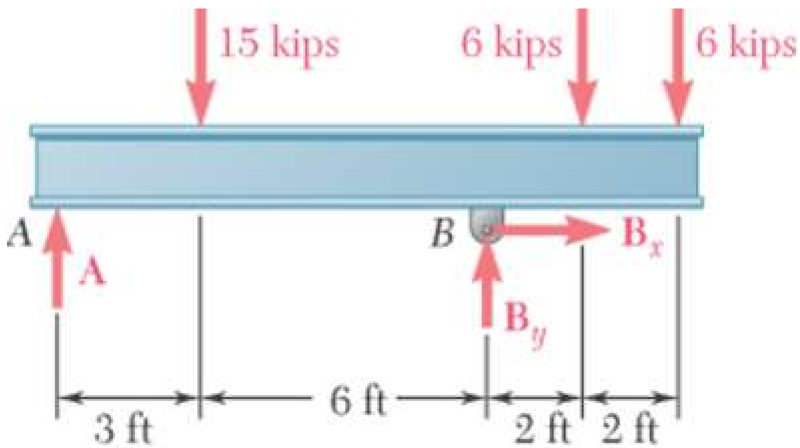
- Check the values obtained.

Sample Problem 4.2



Three loads are applied to a beam as shown. The beam is supported by a roller at A and by a pin at B. Neglecting the weight of the beam, determine the reactions at A and B when P=15 kips

Sample Problem 4.2



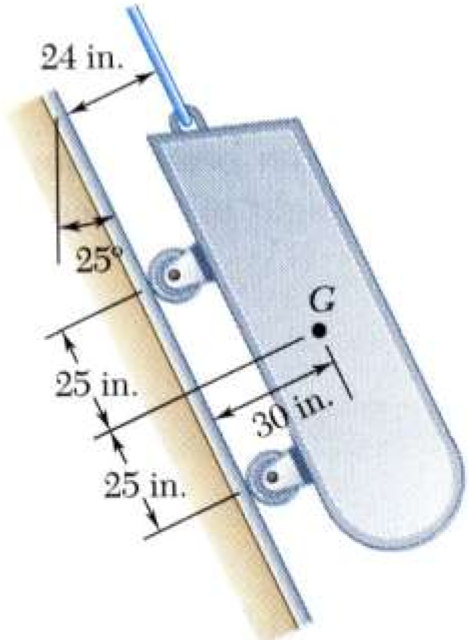
$$\sum \vec{F}_x = 0$$
$$B_x = 0$$

$$\sum \vec{M}_A = 0$$
$$(-15kips)(3\,ft) + B_y(9\,ft) - (6kips)(11\,ft) - (6kips)(13\,ft) = 0$$
$$\boxed{B_y = 21.0kips}$$

$$\sum \vec{M}_B = 0$$
$$-A(9\,ft) + (15kips)(6\,ft) - (6kips)(2\,ft) - (6kips)(4\,ft) = 0$$
$$\boxed{A = 6.00kips}$$

Check ΣFy=0

Sample Problem 4.3



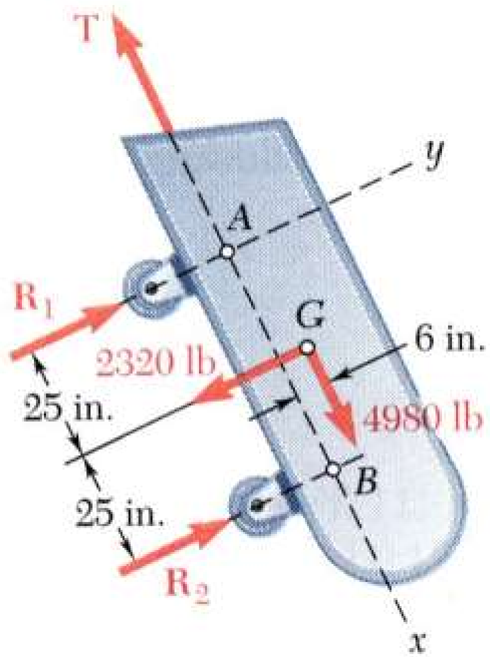
A loading car is at rest on an inclined track. The gross weight of the car and its load is 5500 lb, and it is applied at at *G*. The cart is held in position by the cable.

Determine the tension in the cable and the reaction at each pair of wheels.

SOLUTION:

- Create a free-body diagram for the car with the coordinate system aligned with the track.
- Determine the reactions at the wheels by solving equations for the sum of moments about points above each axle.
- Determine the cable tension by solving the equation for the sum of force components parallel to the track.
- Check the values obtained by verifying that the sum of force components perpendicular to the track are zero.

Sample Problem 4.3



- Create a free-body diagram

$$W_x = +(5500\,\text{lb})\cos 25^\circ$$
$$= +4980\,\text{lb}$$

$$W_y = -(5500\,\text{lb})\sin 25^\circ$$
$$= -2320\,\text{lb}$$

- Determine the reactions at the wheels.

$$\sum M_A = 0: \quad -(2320\,\text{lb})25\text{in.} - (4980\,\text{lb})6\text{in.} + R_2(50\text{in.}) = 0$$

$$\boxed{R_2 = 1758\,\text{lb}}$$

$$\sum M_B = 0: \quad +(2320\,\text{lb})25\text{in.} - (4980\,\text{lb})6\text{in.} - R_1(50\text{in.}) = 0$$

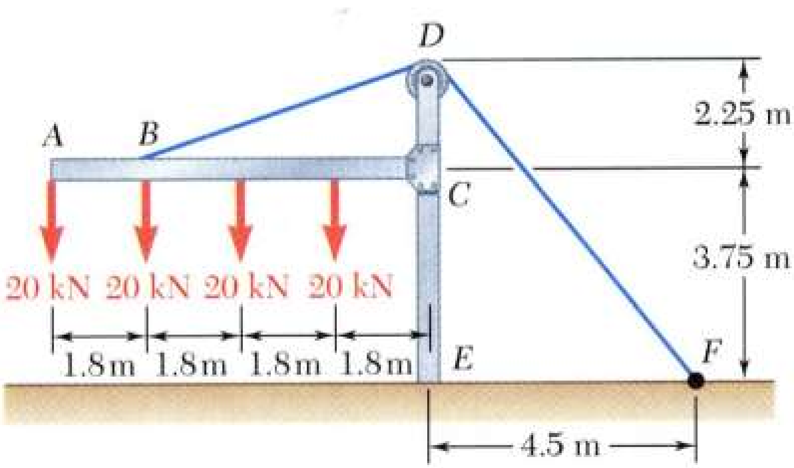
$$\boxed{R_1 = 562\,\text{lb}}$$

- Determine the cable tension.

$$\sum F_x = 0: \quad +4980\,\text{lb} - \text{T} = 0$$

$$\boxed{T = +4980\,\text{lb}}$$

Sample Problem 4.4



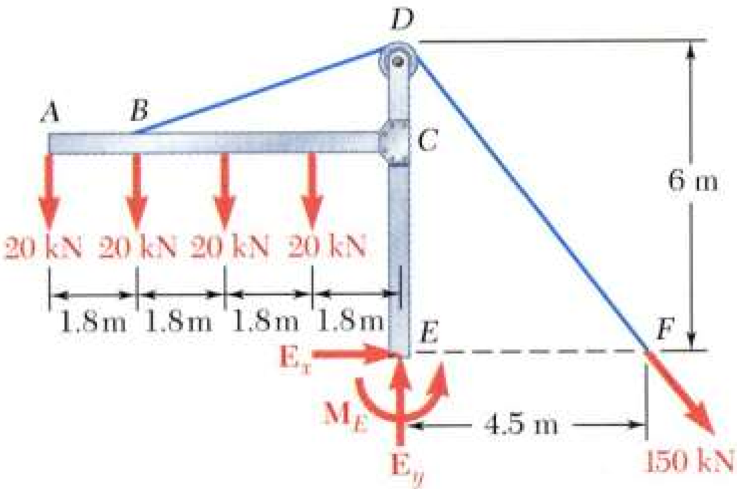
The frame supports part of the roof of a small building. The tension in the cable is 150 kN.

Determine the reaction at the fixed end *E*.

SOLUTION:

- Create a free-body diagram for the frame and cable.
- Solve 3 equilibrium equations for the reaction force components and couple at *E*.

Sample Problem 4.4



- Create a free-body diagram for the frame and cable.

- Solve 3 equilibrium equations for the reaction force components and couple.

$$\sum F_x = 0: \quad E_x + \frac{4.5}{7.5}(150\,\mathrm{kN}) = 0$$

$$\boxed{E_x = -90.0\,\mathrm{kN}}$$

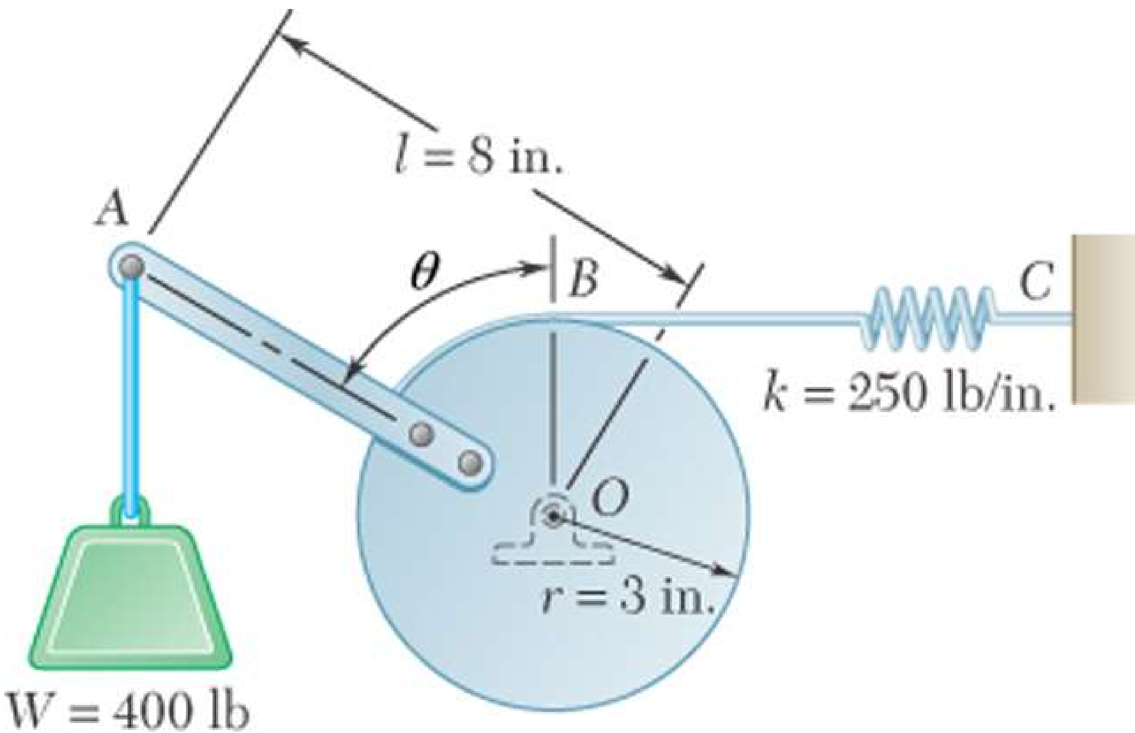
$$\sum F_y = 0: \quad E_y - 4(20\,\mathrm{kN}) - \frac{6}{7.5}(150\,\mathrm{kN}) = 0$$

$$\boxed{E_y = +200\,\mathrm{kN}}$$

$$\sum M_E = 0: \quad +20\,\mathrm{kN}(7.2\,\mathrm{m}) + 20\,\mathrm{kN}(5.4\,\mathrm{m}) + 20\,\mathrm{kN}(3.6\,\mathrm{m}) + 20\,\mathrm{kN}(1.8\,\mathrm{m}) - \frac{6}{7.5}(150\,\mathrm{kN})4.5\,\mathrm{m} + M_E = 0$$

$$\boxed{M_E = 180.0\,\mathrm{kN}\cdot\mathrm{m}}$$

Sample Problem 4.5



A 400-lb weight is attached at A to the lever shown. The constant of the spring BC is k=250 lb/in and the spring is outstretched when f=0. Determine the position of equilibrium

Sample Problem 4.5

$$s = r\theta$$

$$F = ks = kr\theta$$

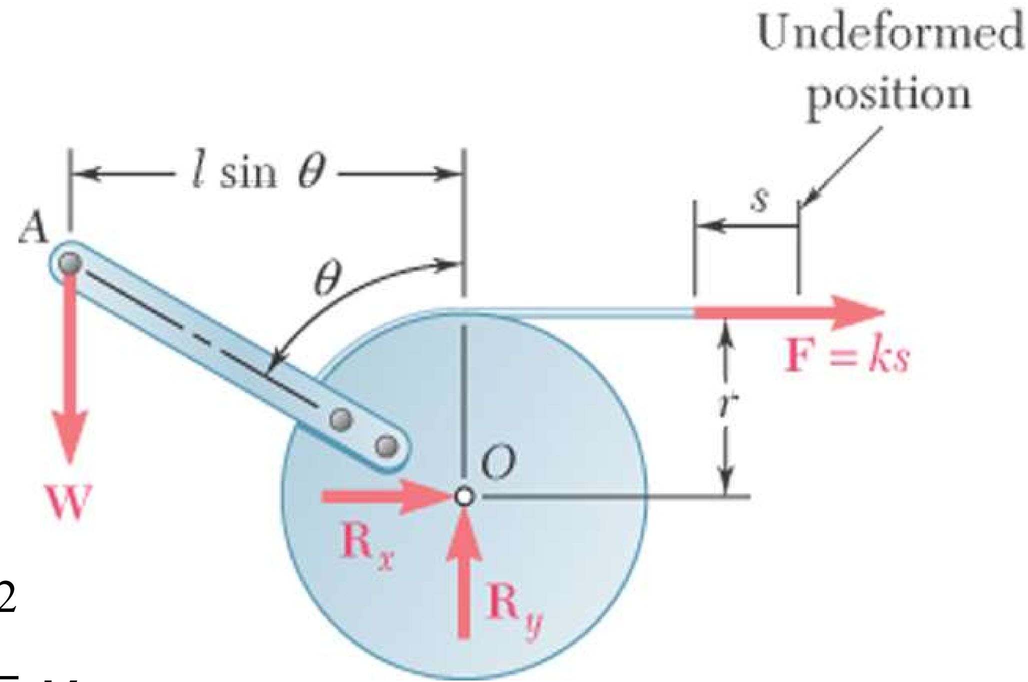
$$\sum M_0 = 0$$

$$Wl\sin\theta - r(kr\theta) = 0$$

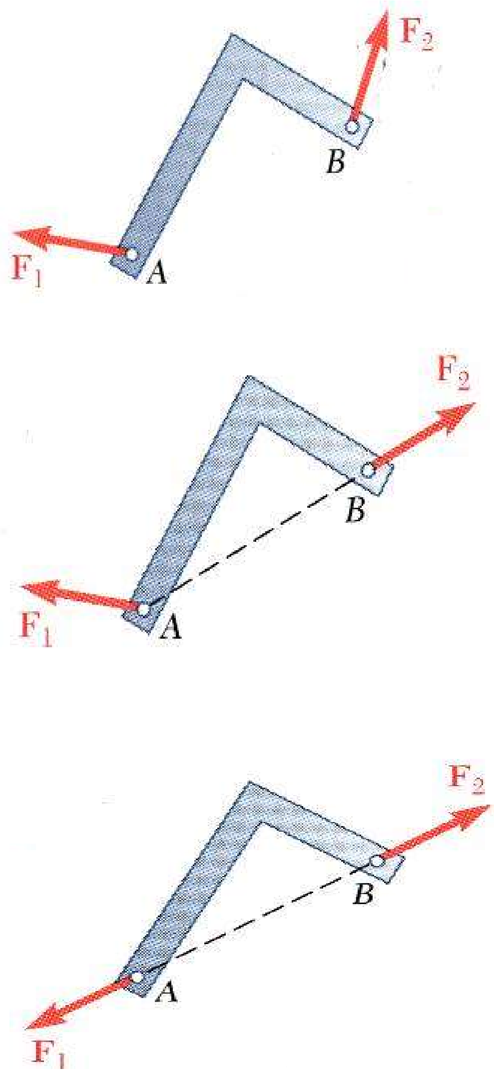
$$\sin\theta = \frac{kr^2}{Wl}\theta = \frac{250*3^2}{400*8}\theta$$

$$\sin\theta = 0.703\theta$$

$$\theta = 80.3^0, \quad 0^0$$

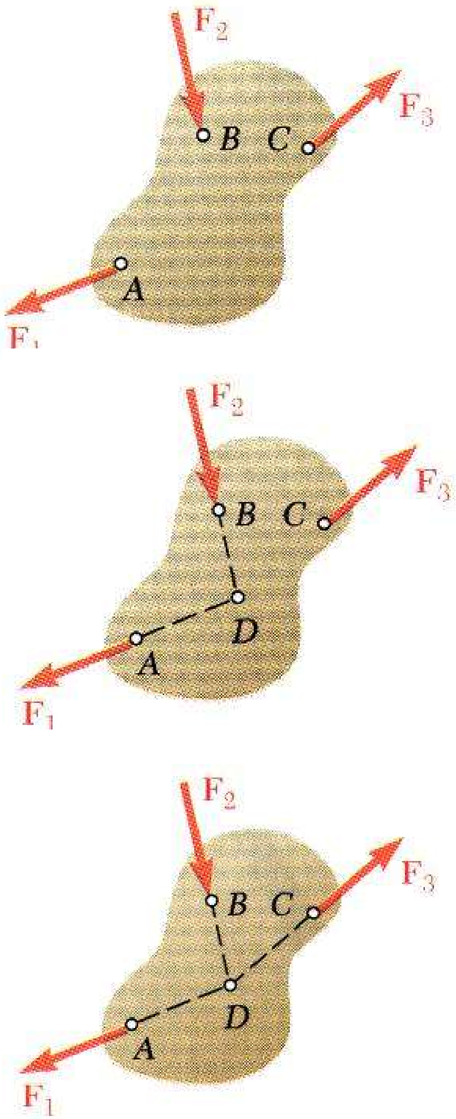


Equilibrium of a Two-Force Body



- Consider a plate subjected to two forces F_1 and F_2
- For static equilibrium, the sum of moments about A must be zero. The moment of F_2 must be zero. It follows that the line of action of F_2 must pass through A.
- Similarly, the line of action of F_1 must pass through B for the sum of moments about B to be zero.
- Requiring that the sum of forces in any direction be zero leads to the conclusion that F_1 and F_2 must have equal magnitude but opposite sense.

Equilibrium of a Three-Force Body



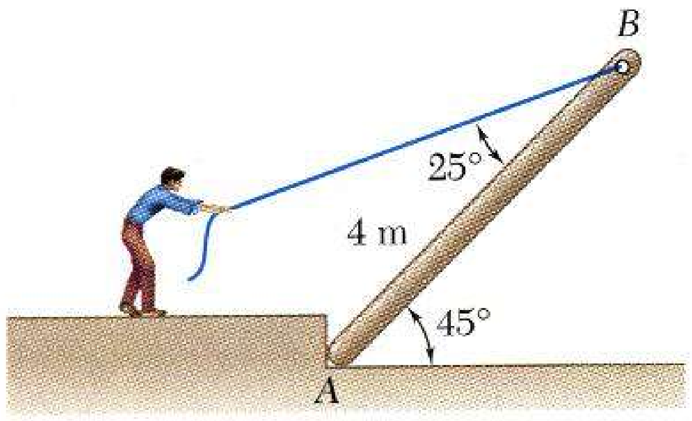
- Consider a rigid body subjected to forces acting at only 3 points.

- Assuming that their lines of action intersect, the moment of F_1 and F_2 about the point of intersection represented by D is zero.

- Since the rigid body is in equilibrium, the sum of the moments of F_1, F_2, and F_3 about any axis must be zero. It follows that the moment of F_3 about D must be zero as well and that the line of action of F_3 must pass through D.

- The lines of action of the three forces must be concurrent or parallel.

Sample Problem 4.6



A man raises a 10 kg joist, of length 4 m, by pulling on a rope.

Find the tension in the rope and the reaction at *A*.

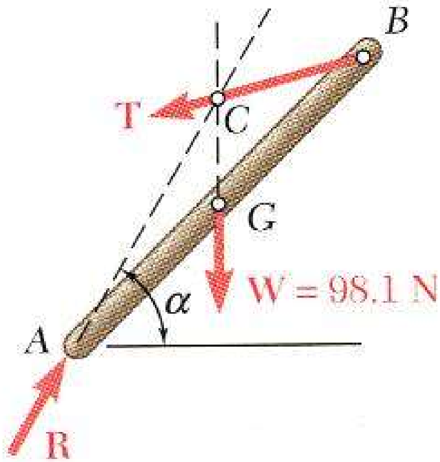
SOLUTION:

- Create a free-body diagram of the joist. Note that the joist is a 3 force body acted upon by the rope, its weight, and the reaction at *A*.

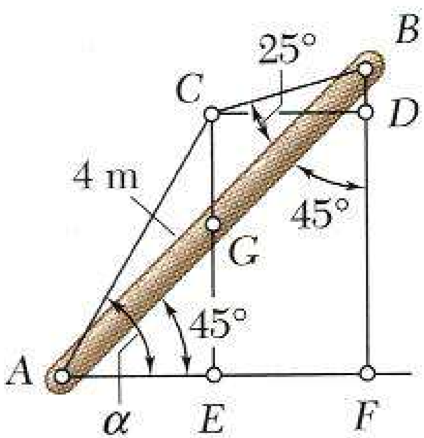
- The three forces must be concurrent for static equilibrium. Therefore, the reaction *R* must pass through the intersection of the lines of action of the weight and rope forces. Determine the direction of the reaction force *R*.

- Utilize a force triangle to determine the magnitude of the reaction force *R*.

Sample Problem 4.6



- Create a free-body diagram of the joist.
- Determine the direction of the reaction force R.



$$AF = AB\cos 45 = (4\,\mathrm{m})\cos 45 = 2.828\,\mathrm{m}$$

$$CD = AE = \tfrac{1}{2}AF = 1.414\ \mathrm{m}$$

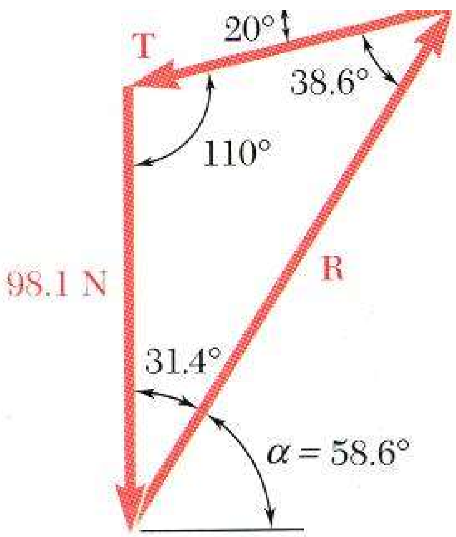
$$BD = CD\cot(45+25) = (1.414\ \mathrm{m})\tan 20 = 0.515\ \mathrm{m}$$

$$CE = BF - BD = (2.828 - 0.515)\ \mathrm{m} = 2.313\ \mathrm{m}$$

$$\tan\alpha = \frac{CE}{AE} = \frac{2.313}{1.414} = 1.636$$

$$\boxed{\alpha = 58.6^\circ}$$

Sample Problem 4.6

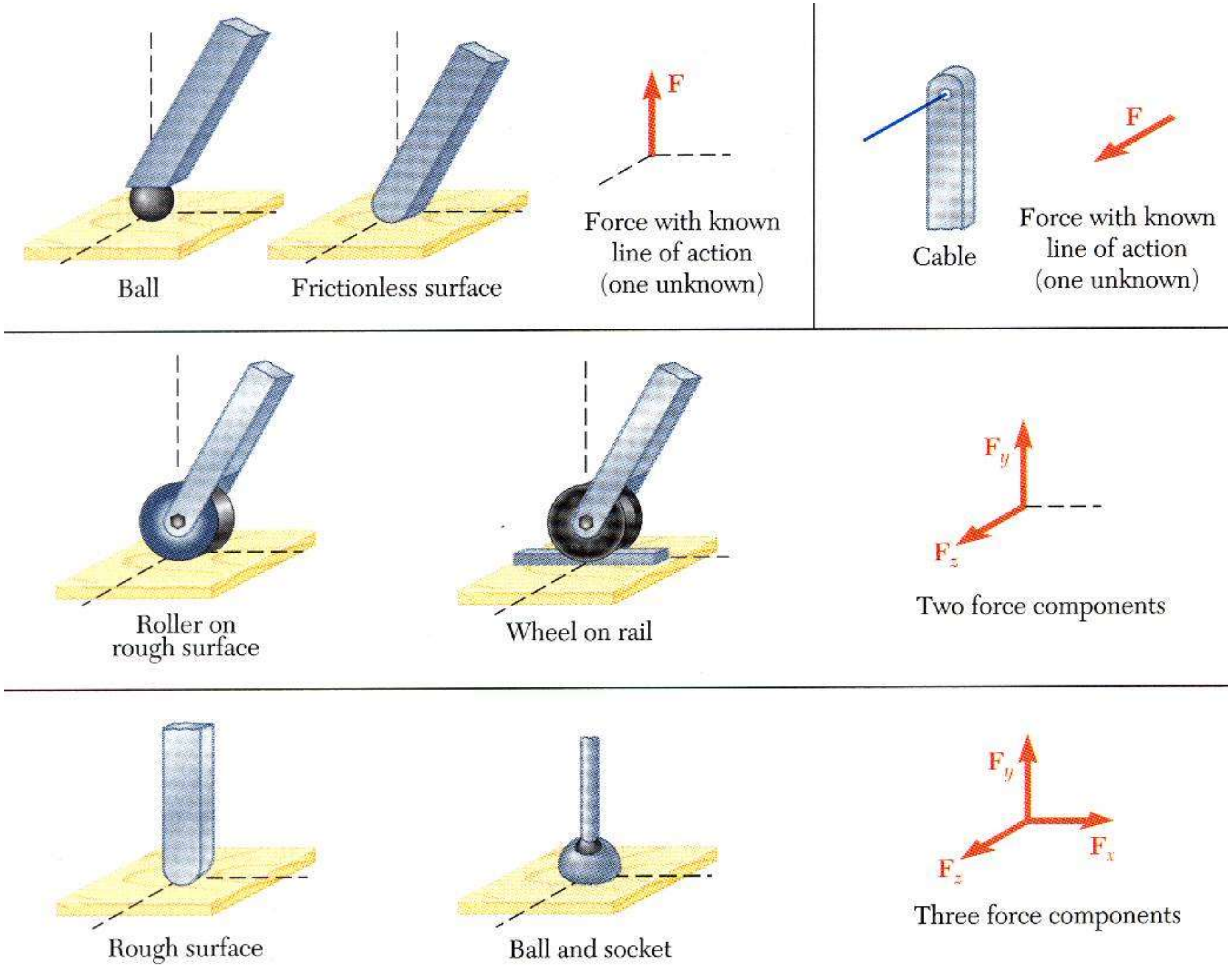


- Determine the magnitude of the reaction force R.

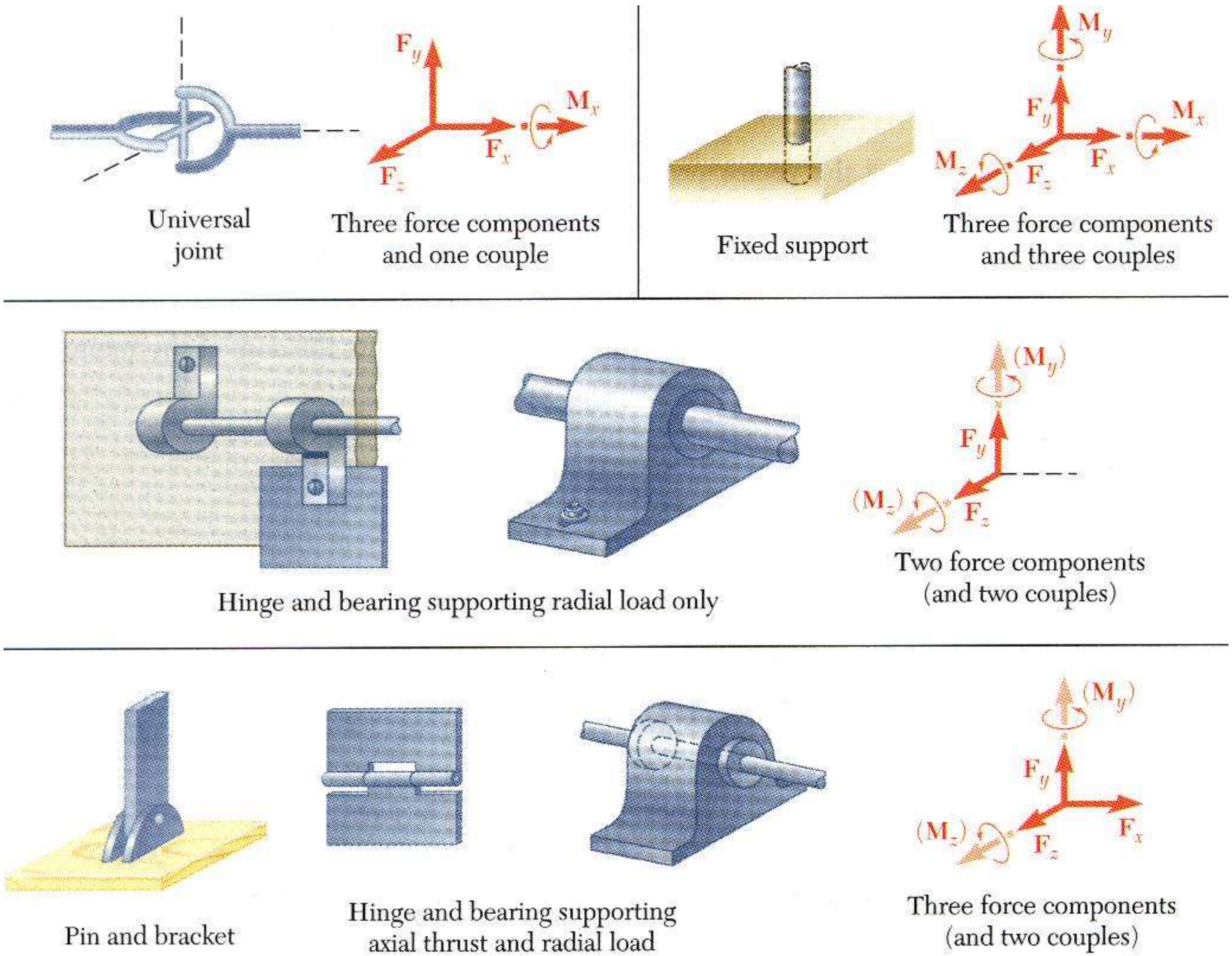
$$\frac{T}{\sin 31.4^\circ} = \frac{R}{\sin 110^\circ} = \frac{98.1\,\text{N}}{\sin 38.6^\circ}$$

$$\boxed{\begin{aligned} T &= 81.9\,\text{N} \\ R &= 147.8\,\text{N} \end{aligned}}$$

Reactions at Supports and Connections for a Three-Dimensional Structure



Reactions at Supports and Connections for a Three-Dimensional Structure

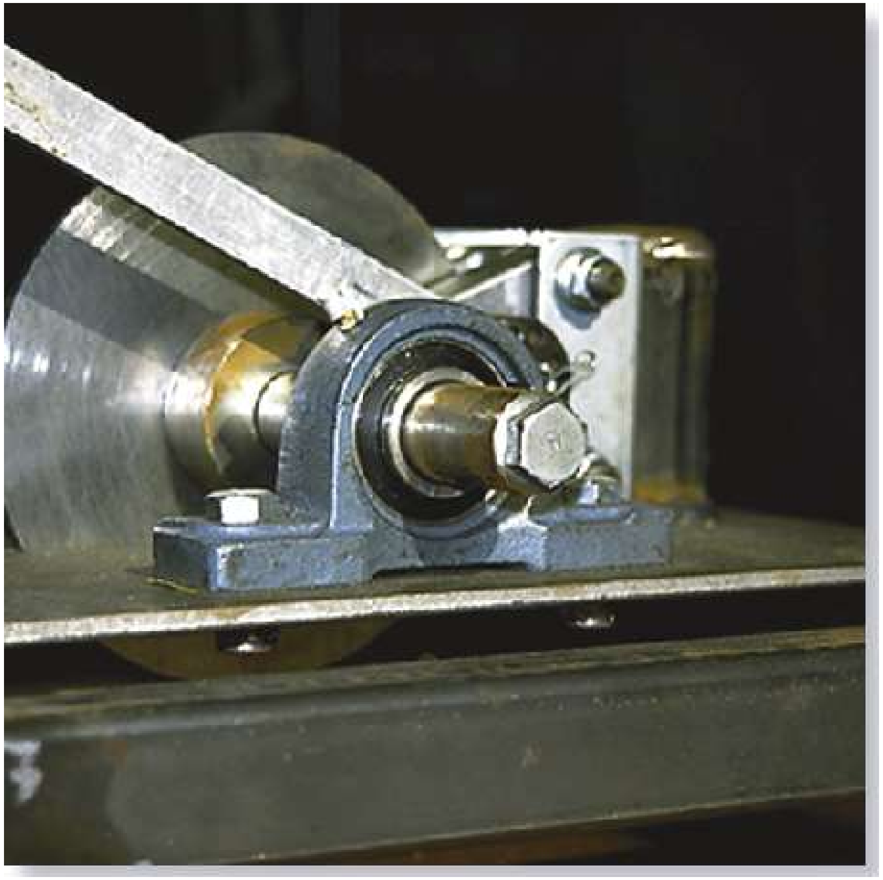


Examples of supports

Examples of supports

This ball-and-socket joint provides a connection for the housing of an earth grader to its frame (reference: Statics, 9th Edition, Hibbeler).

This journal bearing supports the end of a shaft (reference: Statics, 9th Edition, Hibbeler).



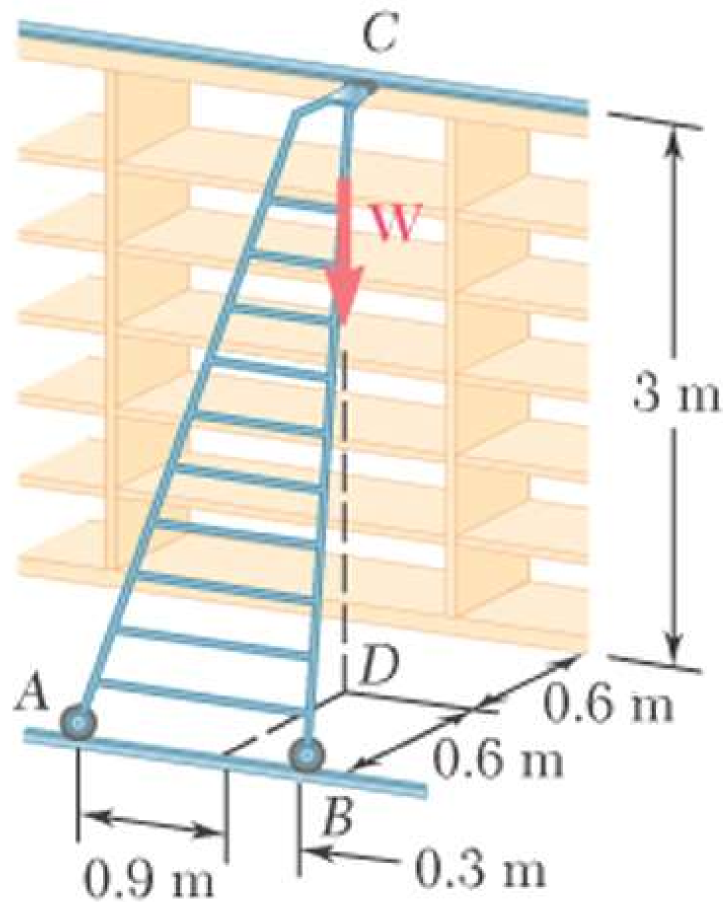
Examples of supports

This thrust bearing is used to support the drive shaft on a machine (reference: Statics, 9th Edition, Hibbeler).

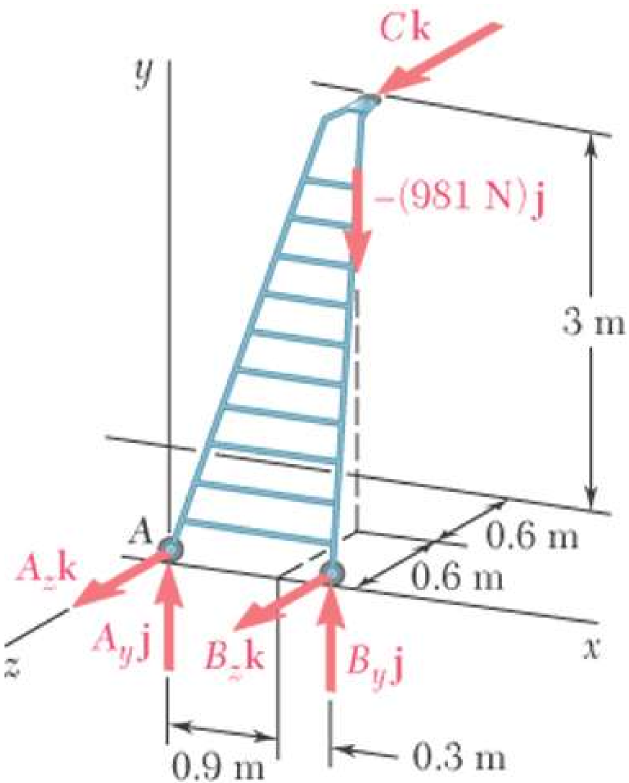
This pin is used to support the end of the strut used on a tractor (reference: Statics, 9th Edition, Hibbeler).

Sample problem 4.7

- A 20-kg ladder used to reach high shelves storeroom is supported by two flanged wl and B mounted on a rail and by an unflan wheel C resting against a rail fixed to the An 80-kg man stands on the ladder and le the right. The line of action of the combin weight W of the man and ladder intersects floor at point D. Determine the reactions a B, and C.



Sample problem 4.7



$$W = -mg\vec{j} = -(80+20)(9.81)\vec{j} = -981\vec{j}$$

$$\sum F = 0$$

$$A_y\vec{j} + A_z\vec{k} + B_y\vec{j} + B_z\vec{k} - 981\vec{j} + C\vec{k} = 0$$

$$A_y + B_y - 981 = 0$$

$$A_z + B_z + C = 0$$

$$\sum M_A = 0$$

Sample problem 4.7

$$W = -m g\vec{j} = -(80+20)(9.81)\vec{j} = -981\vec{j}$$

$$A_y + B_y - 981 = 0$$

$$A_z + B_z + C = 0$$

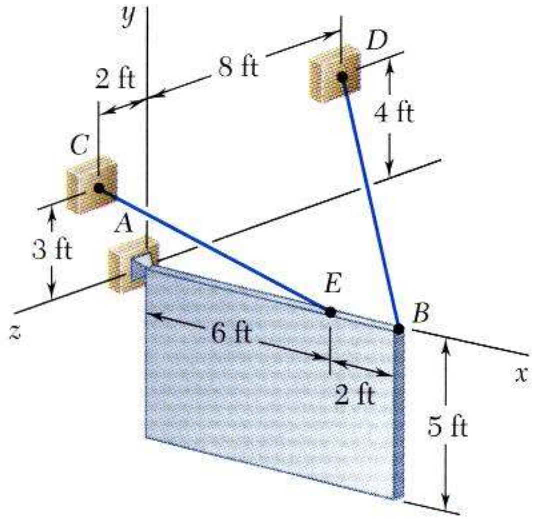
$$\sum M_A = 0$$

$$1.2\vec{i} \times \left(B_y\vec{j} + B_z\vec{k}\right) + \left(0.9\vec{i} - 0.6\vec{k}\right) \times \left(-981\vec{j}\right)$$

$$+\left(0.6\vec{i} + 3\vec{j} - 1.2\vec{k}\right) \times \left(C\vec{k}\right) = 0$$

5 unknowns and 5 equations, so the problem can be solved

Sample Problem 4.8



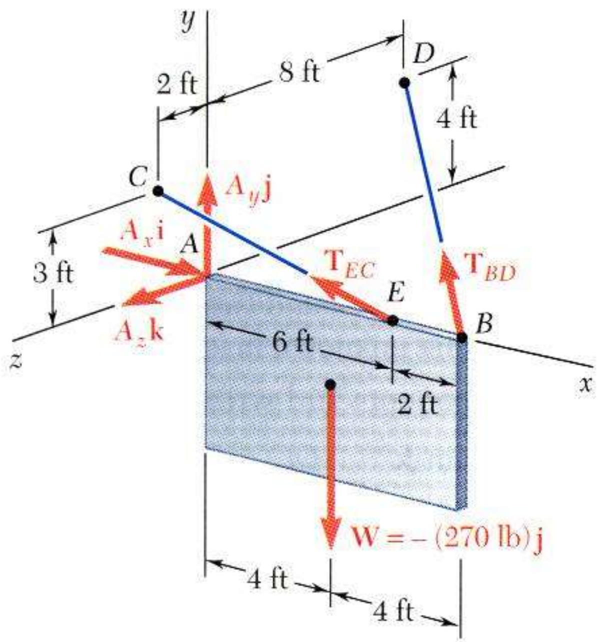
A sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at *A* and by two cables.

Determine the tension in each cable and the reaction at *A*.

SOLUTION:

- Create a free-body diagram for the sign.
- Apply the conditions for static equilibrium to develop equations for the unknown reactions.

Sample Problem 4.8



- Create a free-body diagram for the sign.

  Since there are only 5 unknowns, the sign is partially constrain. It is free to rotate about the *x* axis. It is, however, in equilibrium for the given loading.

$$\vec{T}_{BD} = T_{BD}\frac{\vec{r}_D - \vec{r}_B}{\left|\vec{r}_D - \vec{r}_B\right|}$$

$$= T_{BD}\frac{-8\vec{i} + 4\vec{j} - 8\vec{k}}{12}$$

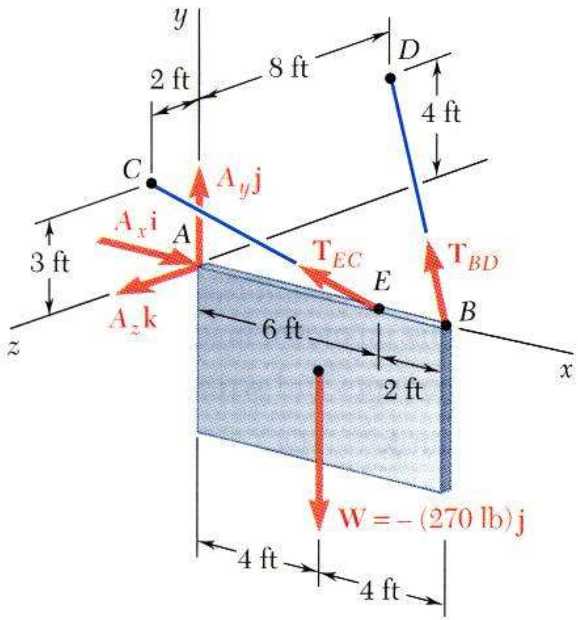
$$= T_{BD}\left(-\tfrac{2}{3}\vec{i} + \tfrac{1}{3}\vec{j} - \tfrac{2}{3}\vec{k}\right)$$

$$\vec{T}_{EC} = T_{EC}\frac{\vec{r}_C - \vec{r}_E}{\left|\vec{r}_C - \vec{r}_E\right|}$$

$$= T_{EC}\frac{-6\vec{i} + 3\vec{j} + 2\vec{k}}{7}$$

$$= T_{EC}\left(-\tfrac{6}{7}\vec{i} + \tfrac{3}{7}\vec{j} + \tfrac{2}{7}\vec{k}\right)$$

Sample Problem 4.8



- Apply the conditions for static equilibrium to develop equations for the unknown reactions.

$$\sum \vec{F} = \vec{A} + \vec{T}_{BD} + \vec{T}_{EC} - (270\,\text{lb})\vec{j} = 0$$

$$\vec{i}: \quad A_x - \tfrac{2}{3}T_{BD} - \tfrac{6}{7}T_{EC} = 0$$

$$\vec{j}: \quad A_y + \tfrac{1}{3}T_{BD} + \tfrac{3}{7}T_{EC} - 270\,\text{lb} = 0$$

$$\vec{k}: \quad A_z - \tfrac{2}{3}T_{BD} + \tfrac{2}{7}T_{EC} = 0$$

$$\sum \vec{M}_A = \vec{r}_B \times \vec{T}_{BD} + \vec{r}_E \times \vec{T}_{EC} + (4\,\text{ft})\vec{i} \times (-270\,\text{lb})\vec{j} = 0$$

$$\vec{j}: \quad 5.333T_{BD} - 1.714T_{EC} = 0$$

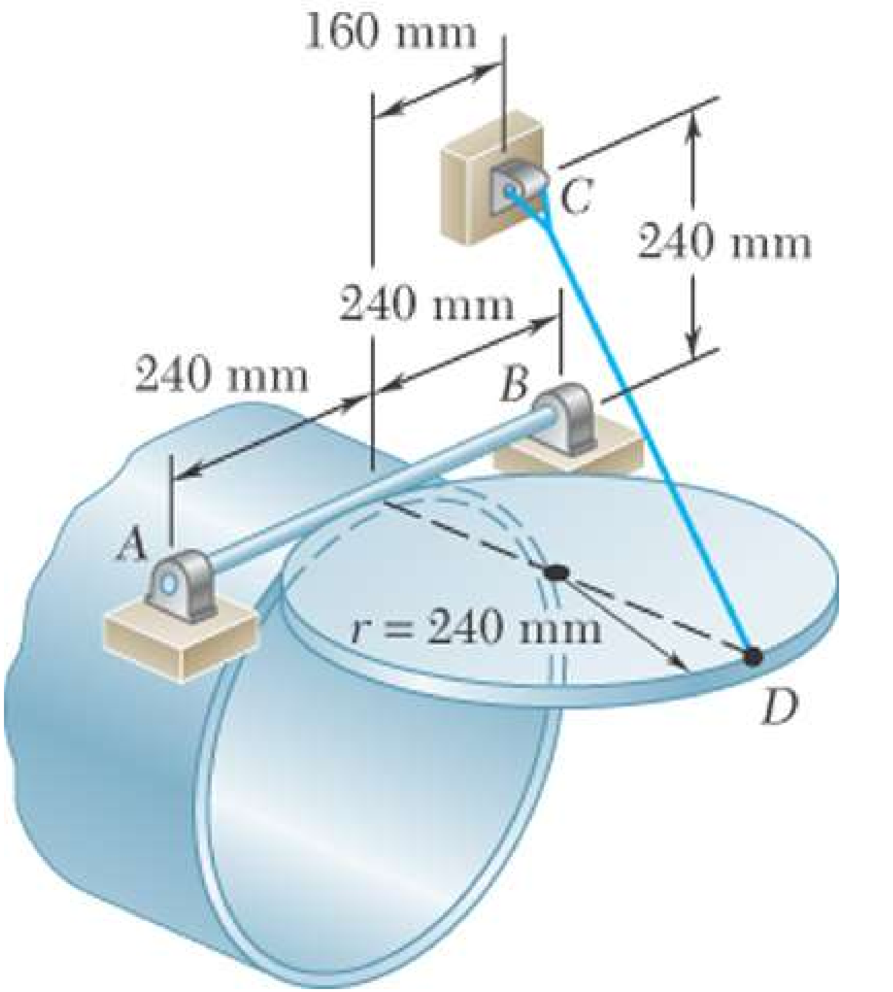
$$\vec{k}: \quad 2.667T_{BD} + 2.571T_{EC} - 1080\,\text{lb} = 0$$

Solve the 5 equations for the 5 unknowns,

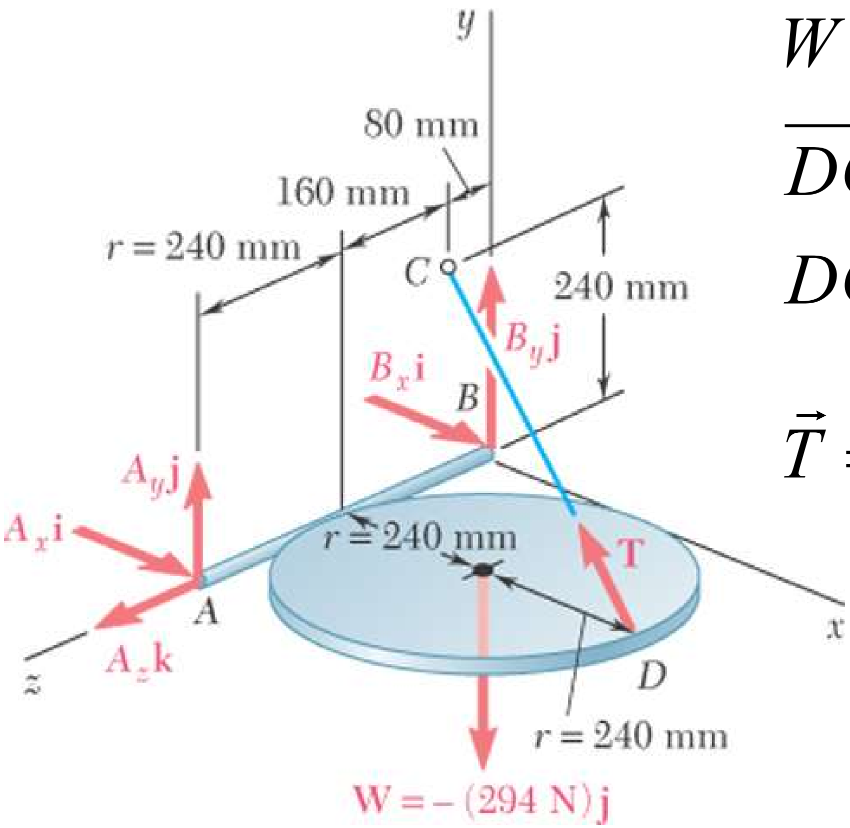
$$\boxed{\begin{array}{l} T_{BD} = 101.3\,\text{lb} \quad T_{EC} = 315\,\text{lb} \\ \vec{A} = (338\,\text{lb})\vec{i} + (101.2\,\text{lb})\vec{j} - (22.5\,\text{lb})\vec{k} \end{array}}$$

Sample Problem 4.9

- A uniform pipe cover of radius r = 240 mm and mass 30 kg is held in a horizontal position by the cable CD. Assuming that the bearing at B does not exert any axial thrust, determine the tension in the cable and the reactions at A and B.



Sample Problem 4.9



$$W = -m g\vec{j} = -(30)(9.81)\vec{j} = -294\vec{j}$$

$$\overrightarrow{DC} = -480\vec{i} + 240\vec{j} - 160\vec{k}$$

$$DC = 560 mm$$

$$\vec{T} = T\frac{\overrightarrow{DC}}{DC} = -\frac{6}{7}\vec{i} + \frac{3}{7}\vec{j} - \frac{2}{7}\vec{k}$$

Sample Problem 4.9

$$\sum F = 0$$

$$A_x\vec{i} + A_y\vec{j} + A_z\vec{k} + B_x\vec{i} + B_y\vec{j} + \vec{T} - 294\vec{j} = 0$$

$$\left(A_x + B_x - \frac{6}{7}T\right)\vec{i} + \left(A_y + B_y + \frac{3}{7}T - 294\right)\vec{j}$$

$$\left(A_z - \frac{2}{7}T\right)\vec{k} = 0$$

6 unknowns and 6 equations

$$\sum M_B = 0$$

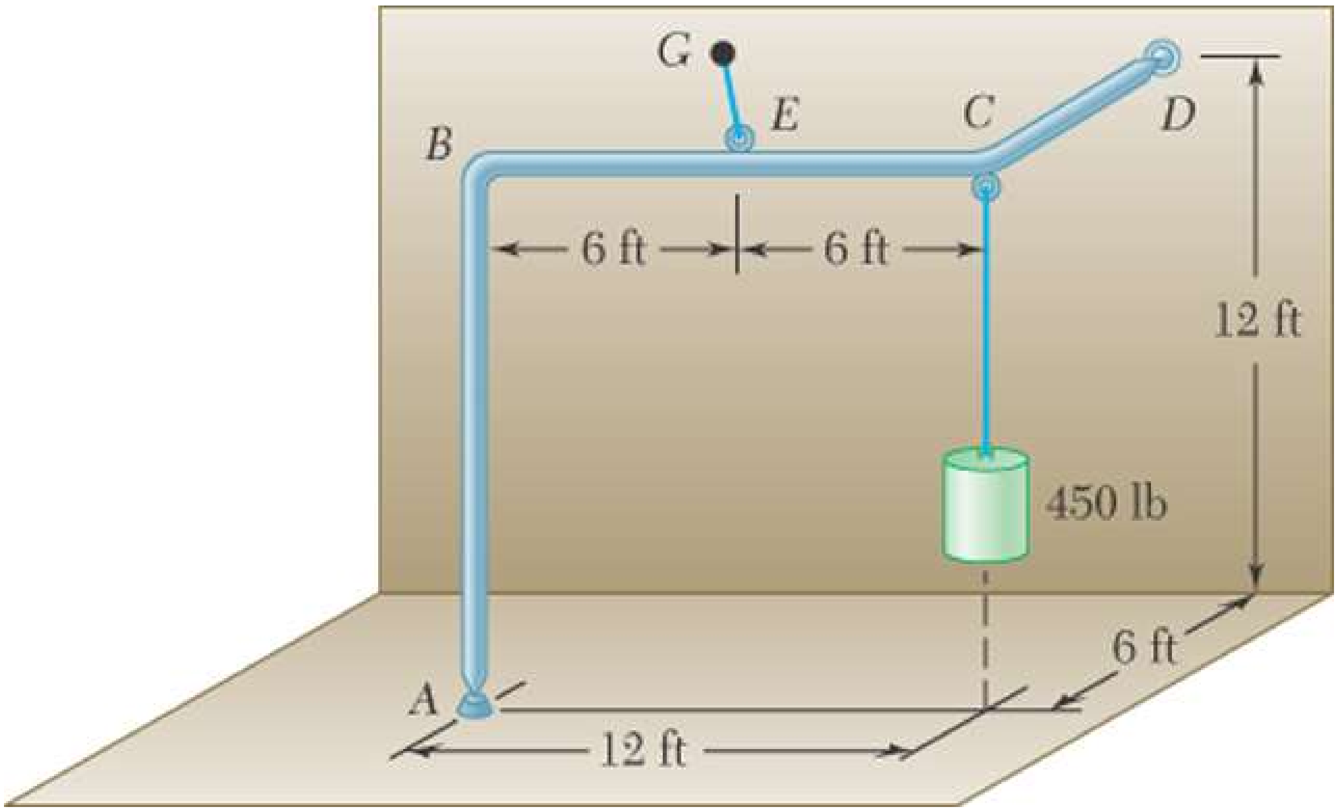
$$2r\vec{k} \times \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right) + \left(2r\vec{i} + r\vec{k}\right) \times \left(-\frac{6}{7}T\vec{i} + \frac{3}{7}T\vec{j} - \frac{2}{7}T\vec{k}\right)$$

$$+\left(r\vec{i} + r\vec{k}\right) \times \left(-294\vec{k}\right) = 0$$

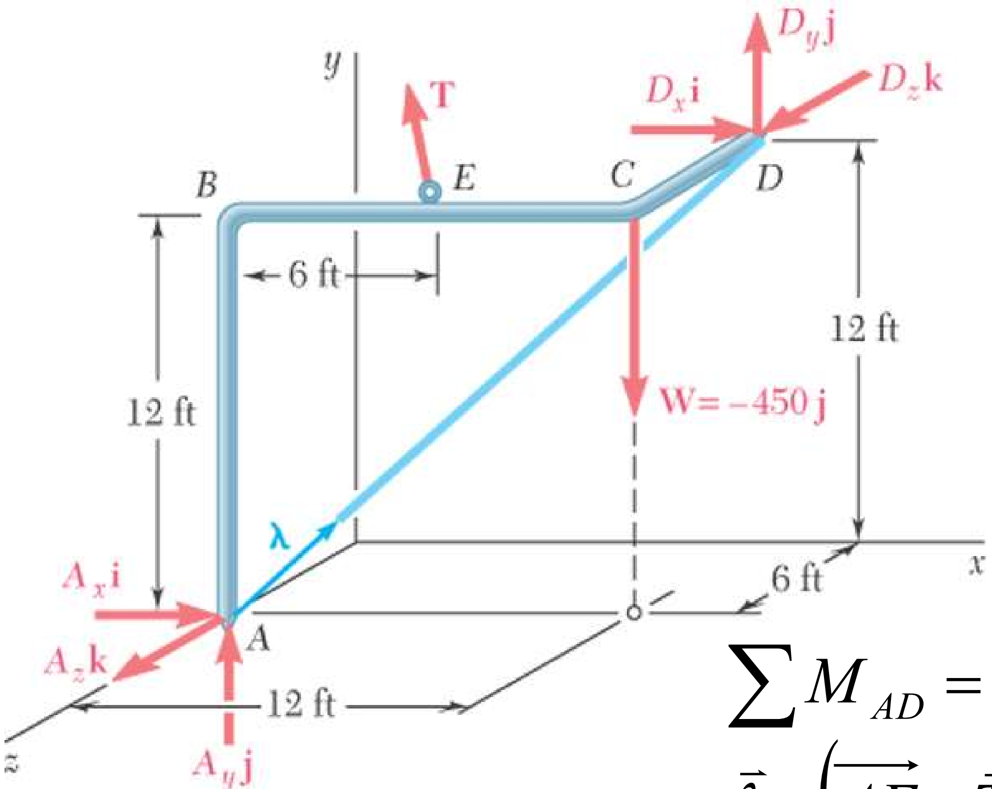
Sample Problem 4.10

- A 450-lb load hangs from the comer C of a rigid piece of pipe ABCD which has been bent as shown. The pipe is supported by the ball-and-socket joints A and D, which are fastened, respectively, to the floor and to a vertical wall, and by a cable attached at the midpoint E of the portion BC of the pipe and at a point C on the wall. Determine (a) where G should be located if the tension in the cable is to be minimum, (b) the corresponding minimum value of the tension.

Sample Problem 4.10



Sample Problem 4.10



$$\sum M_{AD} = 0$$

$$\vec{\lambda} \bullet \left(\overrightarrow{AE} \times \vec{T}\right) + \vec{\lambda} \bullet \left(\overrightarrow{AC} \times \vec{W}\right) = 0$$

$$\overrightarrow{AC} \times \vec{W} = \left(12\vec{i} + 12\vec{j}\right) \times \left(-450\vec{j}\right) = -5400\vec{k}$$

$$\vec{\lambda} = \frac{\overrightarrow{AD}}{AD} = \frac{12\vec{i} + 12\vec{j} - 6\vec{k}}{18}$$

Sample Problem 4.10

$$\vec{\lambda} \bullet \left(\overrightarrow{AE} \times \vec{T}\right) = \left(\frac{12\vec{i} + 12\vec{j} - 6\vec{k}}{18}\right) \bullet \left(-5400\vec{k}\right) = -1800$$

$$\vec{\lambda} \bullet \left(\overrightarrow{AE} \times \vec{T}\right) = \vec{T} \bullet \left(\vec{\lambda} \times \overrightarrow{AE}\right)$$

$$T\left(\frac{12\vec{i} + 12\vec{j} - 6\vec{k}}{18}\right) \bullet \left[\left(\frac{12\vec{i} + 12\vec{j} - 6\vec{k}}{18}\right) \times \left(6\vec{i} + 12\vec{j}\right)\right] = -1800$$

$$6T = -1800$$

$$\boxed{\begin{aligned} &T_{\min} = -200\vec{i} + 100\vec{j} - 200\vec{k} \\ &\overrightarrow{EG} = (x-6)\vec{i} + (y-12)\vec{j} + (0-6)\vec{k} \end{aligned}}$$

Must be the same direction

x=0, y=15 ft

CHAPTER

5

VECTOR MECHANICS FOR ENGINEERS:
STATICS

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Distributed Forces:
Centroids and Centers of Gravity

Contents

Introduction

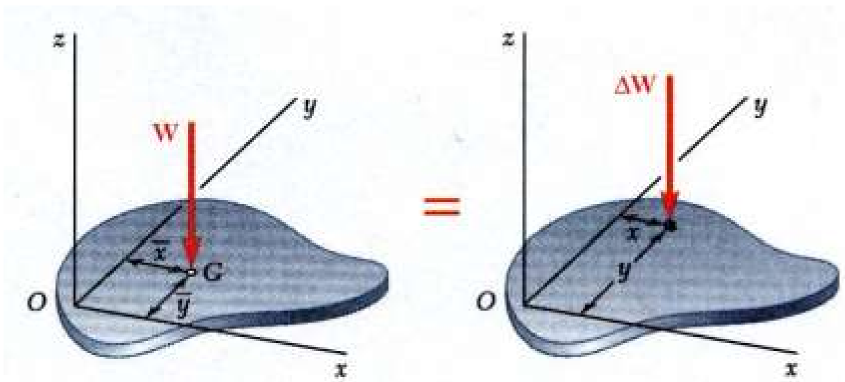
- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.

- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.

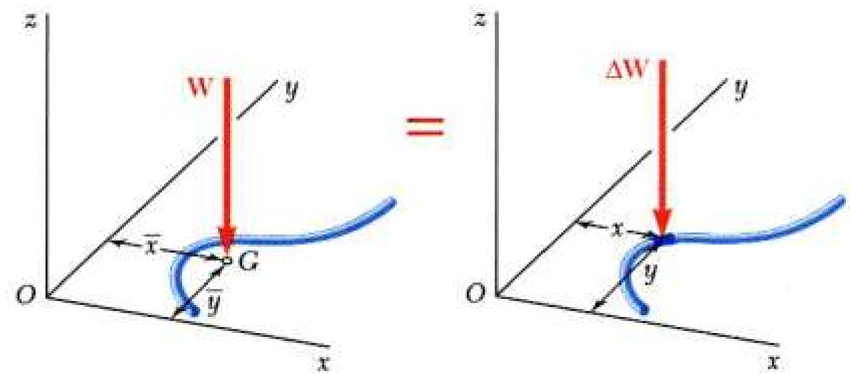
- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

Center of Gravity of a 2D Body

- Center of gravity of a plate



- Center of gravity of a wire



$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$

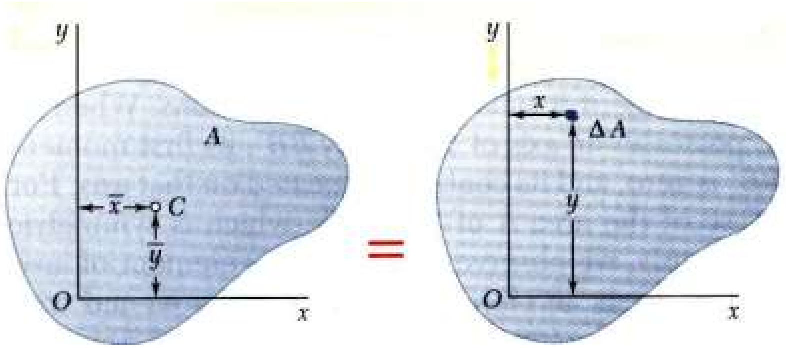
$$= \int x\,dW$$

$$\sum M_y \quad \bar{y}W = \sum y\Delta W$$

$$= \int y\,dW$$

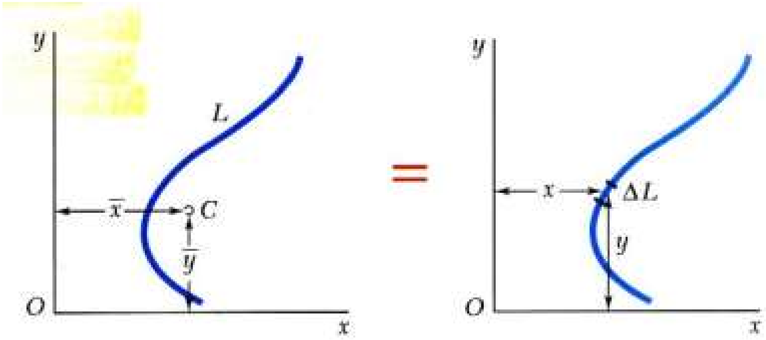
Centroids and First Moments of Areas and Lines

- Centroid of an area



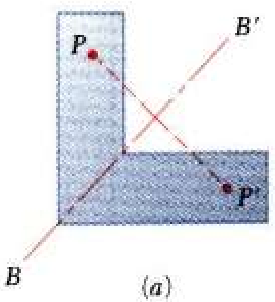
$$\bar{x}W = \int x\,dW$$
$$\bar{x}(\gamma At) = \int x(\gamma t)dA$$
$$\bar{x}A = \int x\,dA = Q_y$$
$$= \text{first moment with respect to } y$$
$$\bar{y}A = \int y\,dA = Q_x$$
$$= \text{first moment with respect to } x$$

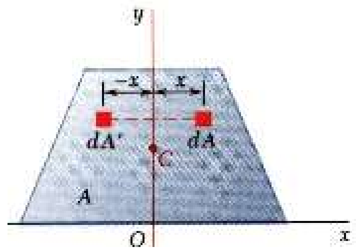
- Centroid of a line

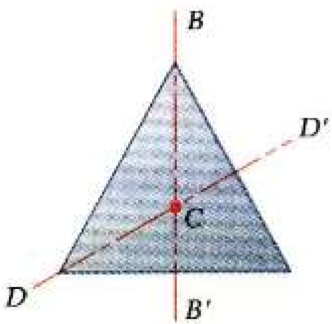


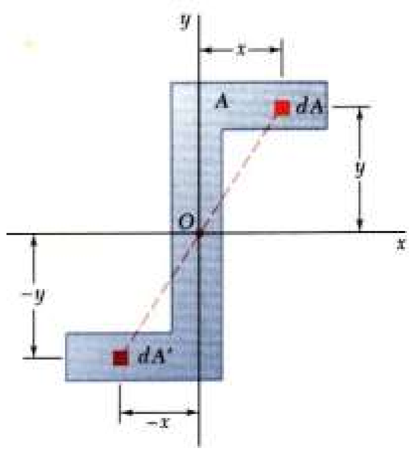
$$\bar{x}W = \int x\,dW$$
$$\bar{x}(\gamma La) = \int x(\gamma a)dL$$
$$\bar{x}L = \int x\,dL$$
$$\bar{y}L = \int y\,dL$$

First Moments of Areas and Lines



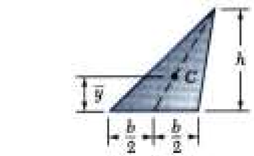
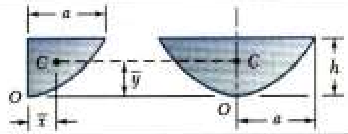
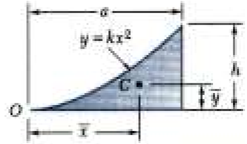
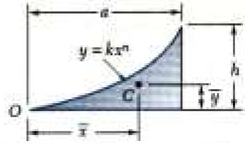
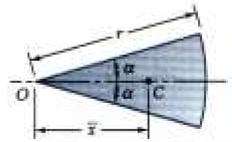




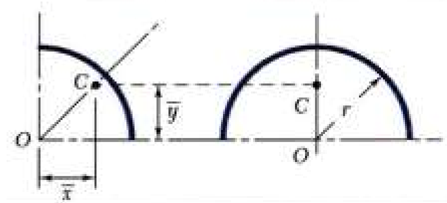
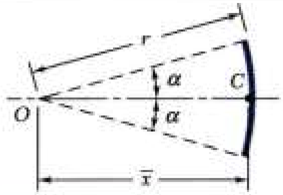


- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.

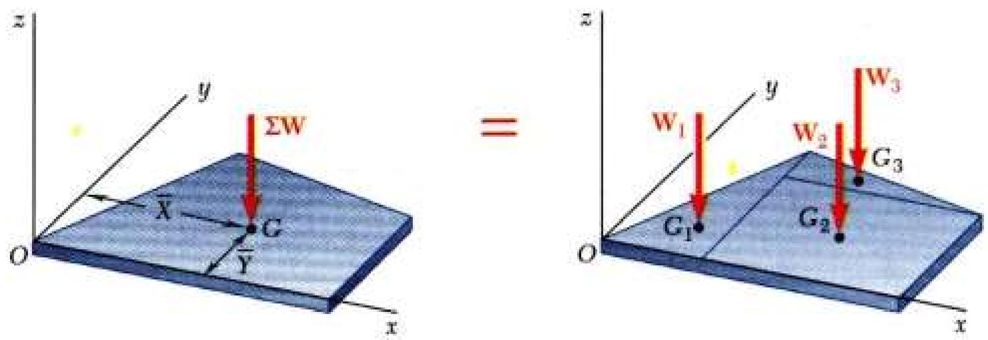
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

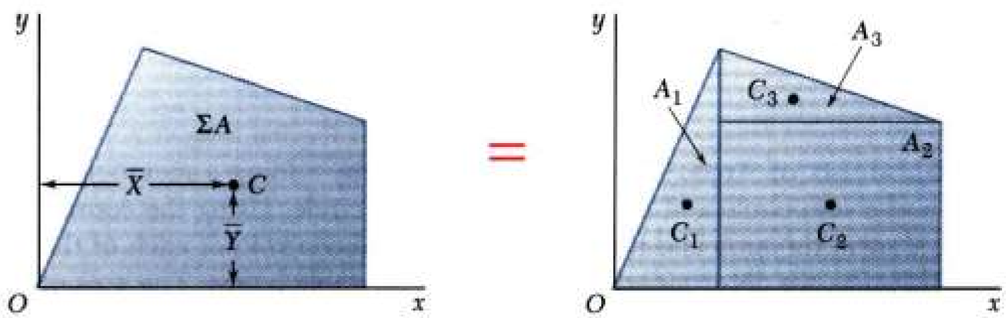
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Composite Plates and Areas



- Composite plates

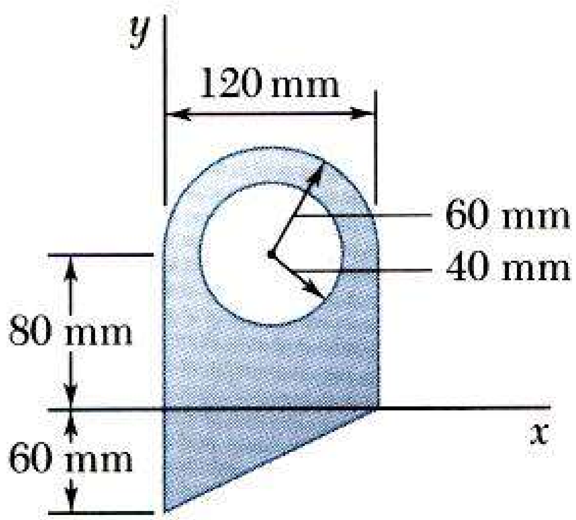
$$\bar{X}\sum W = \sum \bar{x}\,W$$
$$\bar{Y}\sum W = \sum \bar{y}\,W$$



- Composite area

$$\bar{X}\sum A = \sum \bar{x}\,A$$
$$\bar{Y}\sum A = \sum \bar{y}\,A$$

Sample Problem 5.1

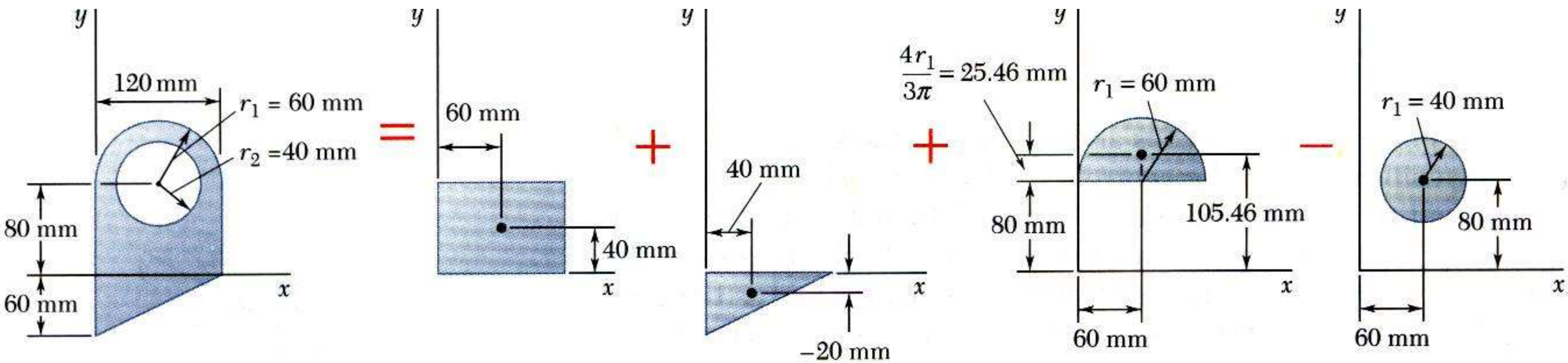


For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem 5.1



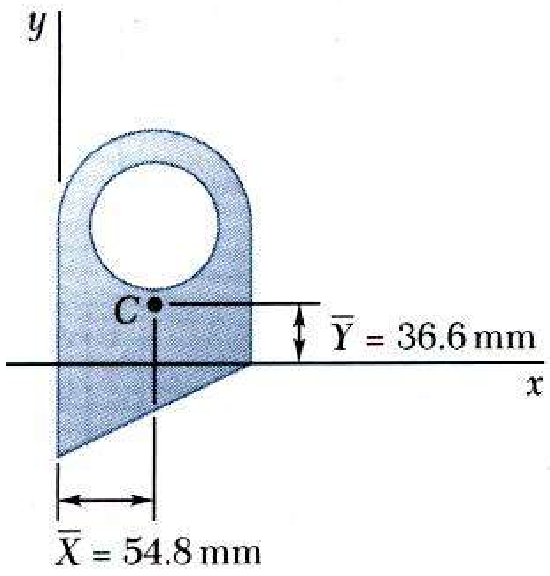
Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$
$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

Sample Problem 5.1

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



$$\overline{X} = \frac{\sum \bar{x} A}{\sum A} = \frac{+757.7 \times 10^3\,\text{mm}^3}{13.828 \times 10^3\,\text{mm}^2}$$

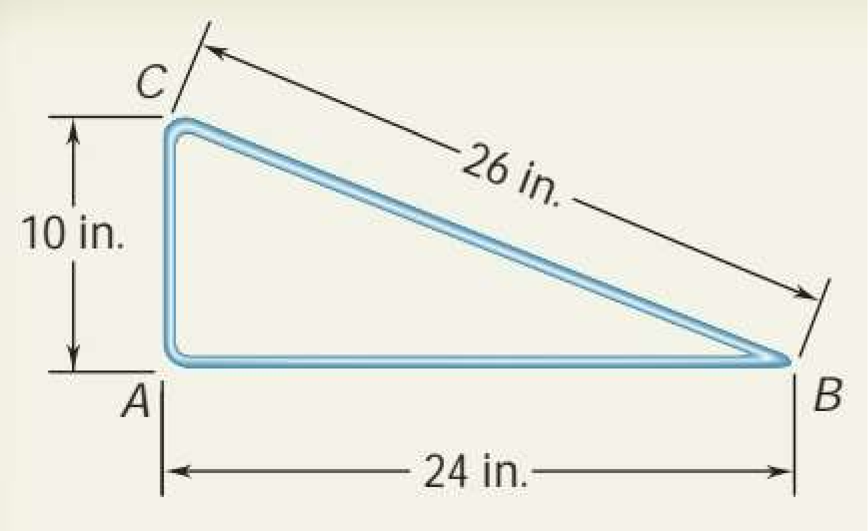
$$\boxed{\overline{X} = 54.8\,\text{mm}}$$

$$\overline{Y} = \frac{\sum \bar{y} A}{\sum A} = \frac{+506.2 \times 10^3\,\text{mm}^3}{13.828 \times 10^3\,\text{mm}^2}$$

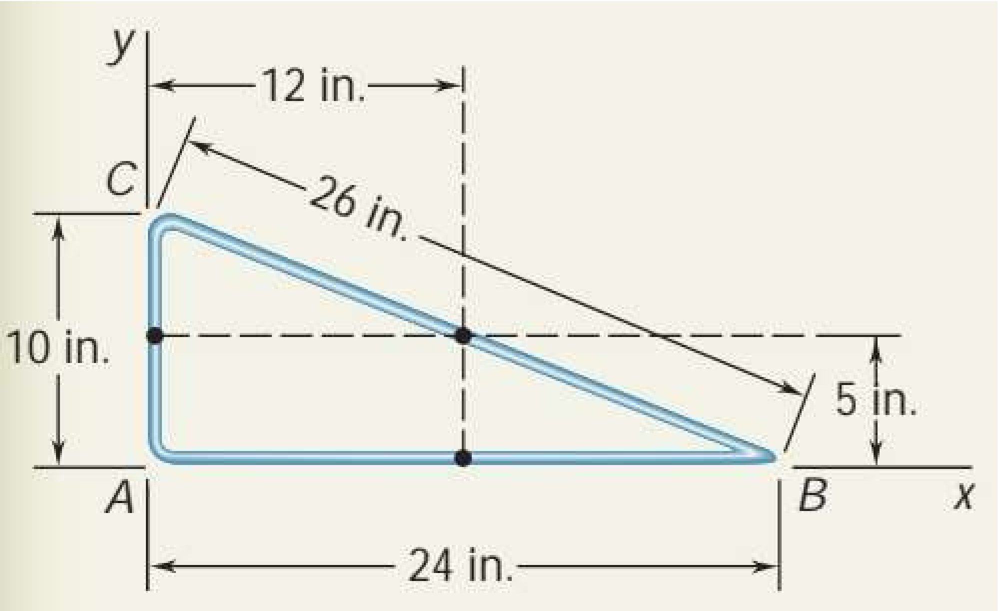
$$\boxed{\overline{Y} = 36.6\,\text{mm}}$$

Sample Problem 5.2

The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.



Sample Problem 5.2



Since the figure is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. Therefore, that centroid will be determined. Choosing the coordinate axes shown, with origin at *A*, we determine the coordinates of the centroid of each line segment and compute the first moments with respect to the coordinate axes.

Sample Problem 5.2

Segment	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
AB	24	12	0	288	0
BC	26	12	5	312	130
CA	10	0	5	0	50
	$\Sigma L = 60$			$\Sigma \bar{x}L = 600$	$\Sigma \bar{y}L = 180$

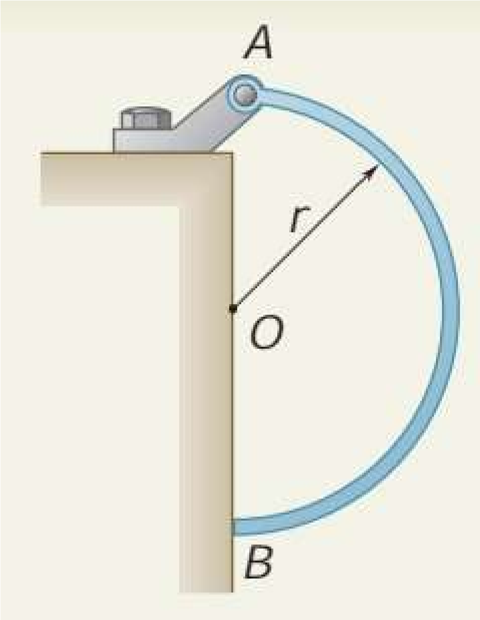
Substituting the values obtained from the table into the equations defining the centroid of a composite line, we obtain

$\bar{X}\Sigma L = \Sigma \bar{x}L$: $\quad \bar{X}(60 \text{ in.}) = 600 \text{ in}^2 \qquad \bar{X} = 10 \text{ in.}$ ◀

$\bar{Y}\Sigma L = \Sigma \bar{y}L$: $\quad \bar{Y}(60 \text{ in.}) = 180 \text{ in}^2 \qquad \bar{Y} = 3 \text{ in.}$ ◀

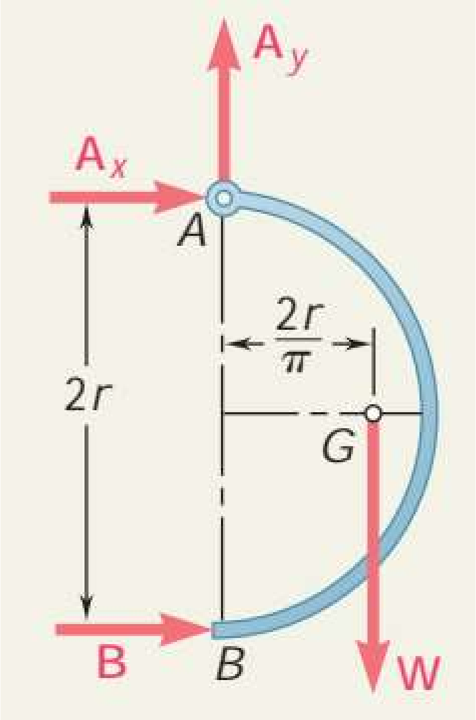
Sample Problem 5.3

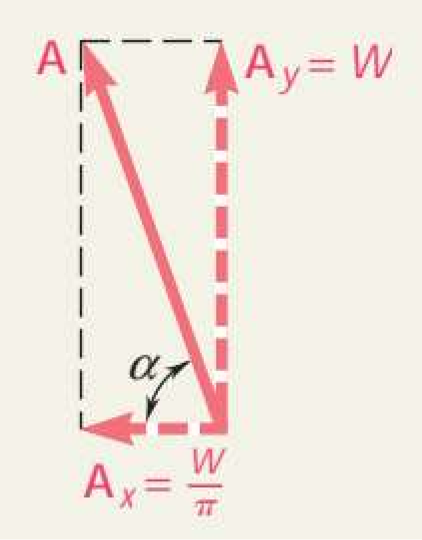
A uniform semicircular rod of weight W and radius r is attached to a pin at A and rests against a frictionless surface at B. Determine the reactions at A and B.



Sample Problem 5.3

Free-Body Diagram. A free-body diagram of the rod is drawn. The forces acting on the rod are its weight **W**, which is applied at the center of gravity G (whose position is obtained from Fig. 5.8B); a reaction at A, represented by its components $\mathbf{A}_x$ and $\mathbf{A}_y$; and a horizontal reaction at B.





Sample Problem 5.3

Equilibrium Equations

$$+\mathrm{l}\ \Sigma M_A = 0: \qquad B(2r) - W\left(\frac{2r}{\mathrm{p}}\right) = 0$$

$$B = +\frac{W}{\mathrm{p}} \qquad\qquad \mathbf{B} = \frac{W}{\mathrm{p}}\mathrm{y}$$

$$\overset{+}{\text{ý}}\ \Sigma F_x = 0: \qquad A_x + B = 0$$

$$A_x = -B = -\frac{W}{\mathrm{p}} \qquad \mathbf{A}_x = \frac{W}{\mathrm{p}}\mathrm{z}$$

$$+\mathrm{x}\,\Sigma F_y = 0: \qquad A_y - W = 0 \qquad \mathbf{A}_y = W\mathrm{x}$$

Adding the two components of the reaction at A:

$$A = \left[W^2 + \left(\frac{W}{p}\right)^2\right]^{1/2} \qquad A = W\left(1 + \frac{1}{p^2}\right)^{1/2}$$

$$\tan a = \frac{W}{W/p} = p \qquad a = \tan^{-1} p$$

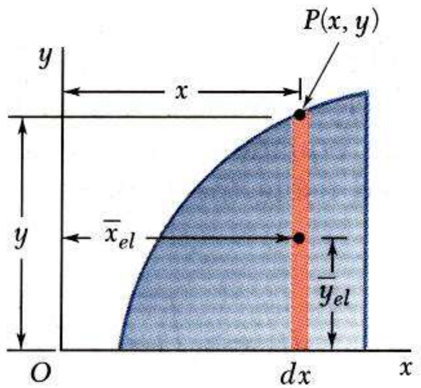
The answers can also be expressed as follows:

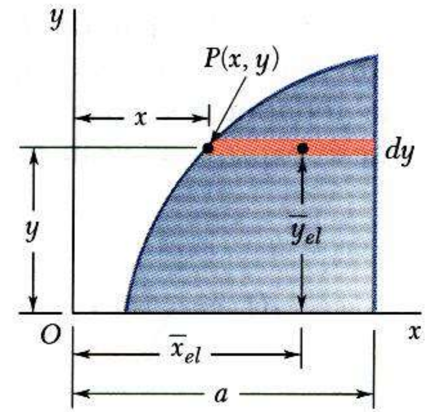
$$\mathbf{A} = 1.049W \text{ b } 72.3° \qquad \mathbf{B} = 0.318W\text{y}$$

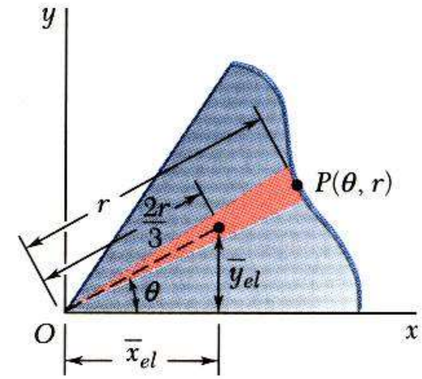
Determination of Centroids by Integration

$$\bar{x}A = \int x dA = \iint x\, dx dy = \int \bar{x}_{el}\, dA$$
$$\bar{y}A = \int y dA = \iint y\, dx dy = \int \bar{y}_{el}\, dA$$

- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.





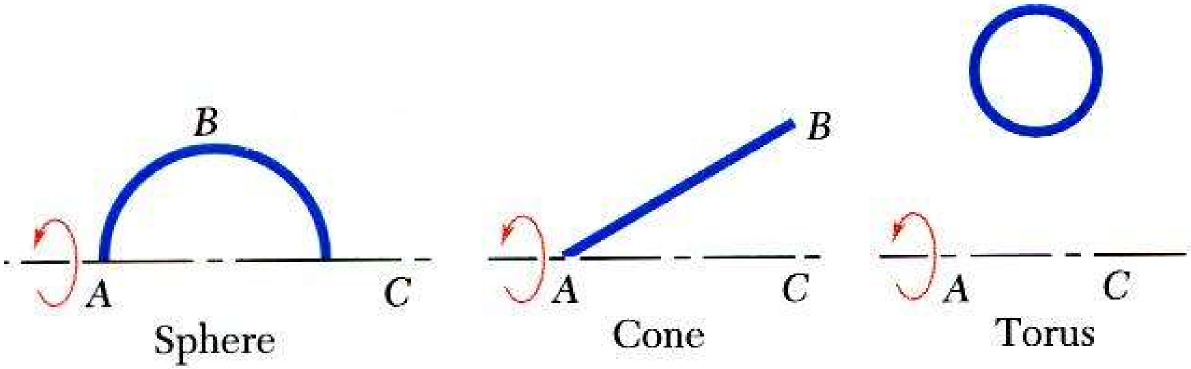


$$\bar{x}A = \int \bar{x}_{el}\, dA = \int x\left(y dx\right)$$
$$\bar{y}A = \int \bar{y}_{el}\, dA = \int \frac{y}{2}\left(y dx\right)$$

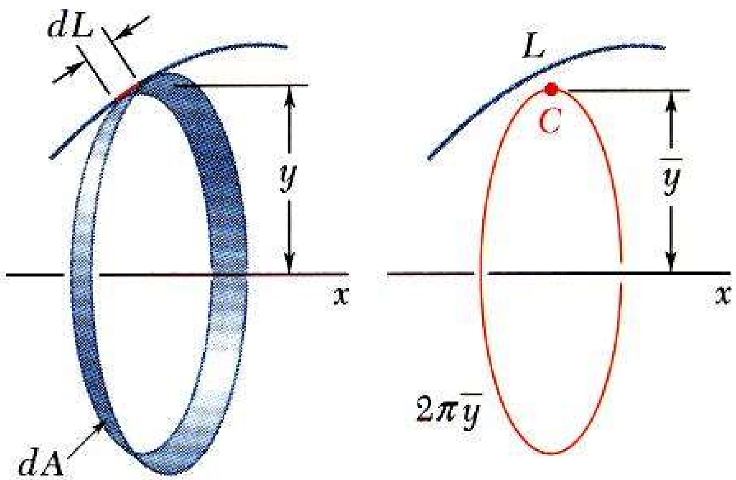
$$\bar{x}A = \int \bar{x}_{el}\, dA = \int \frac{a+x}{2}\left[(a-x)dx\right]$$
$$\bar{y}A = \int \bar{y}_{el}\, dA = \int y\left[(a-x)dx\right]$$

$$\bar{x}A = \int \bar{x}_{el}\, dA = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2 d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\, dA = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2 d\theta\right)$$

Theorems of Pappus-Guldinus



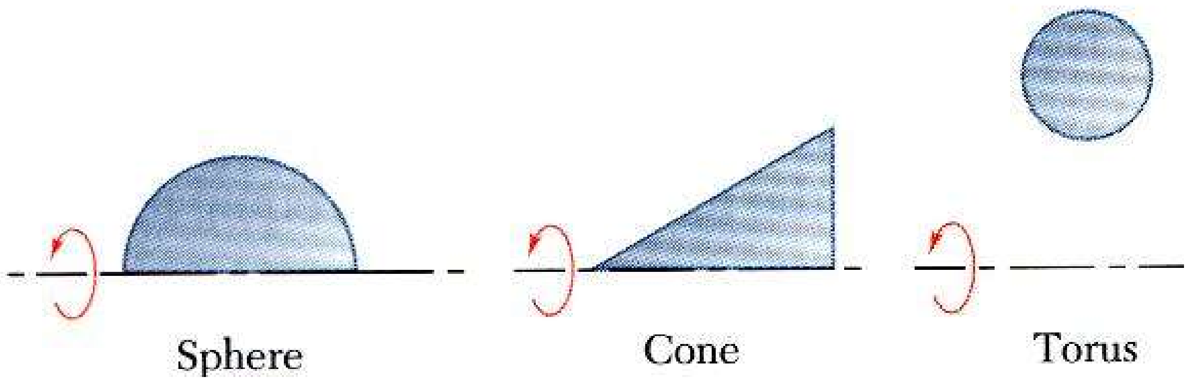
- Surface of revolution is generated by rotating a plane curve about a fixed axis.



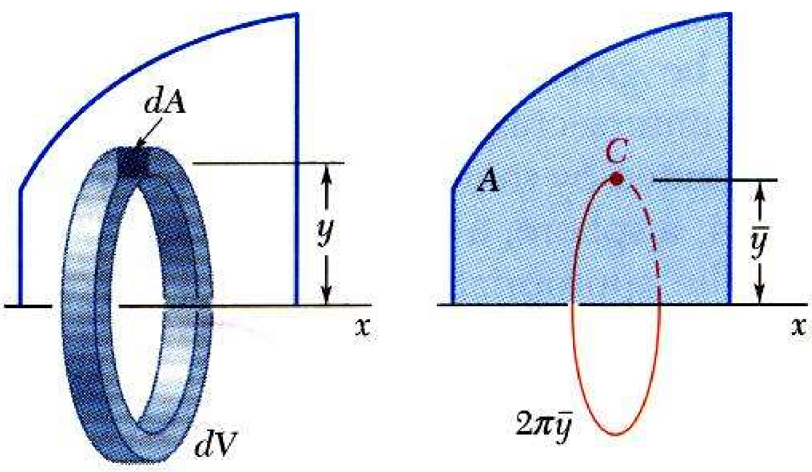
- Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$A = 2\pi \bar{y} L$$

Theorems of Pappus-Guldinus



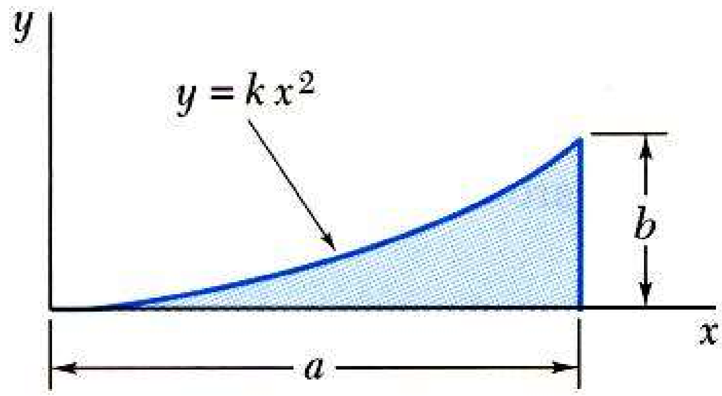
- Body of revolution is generated by rotating a plane area about a fixed axis.



- Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$V = 2\pi \bar{y} A$$

Sample Problem 5.4

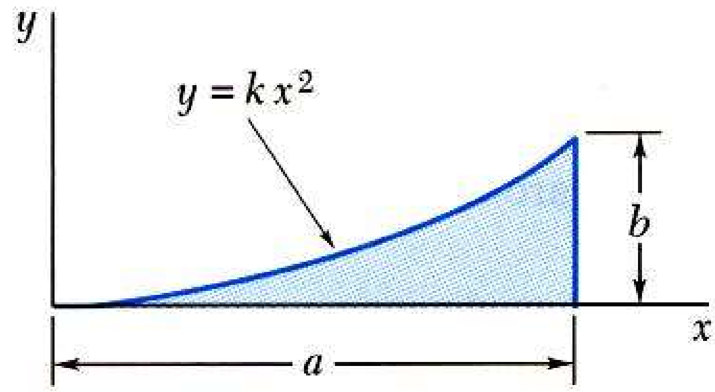


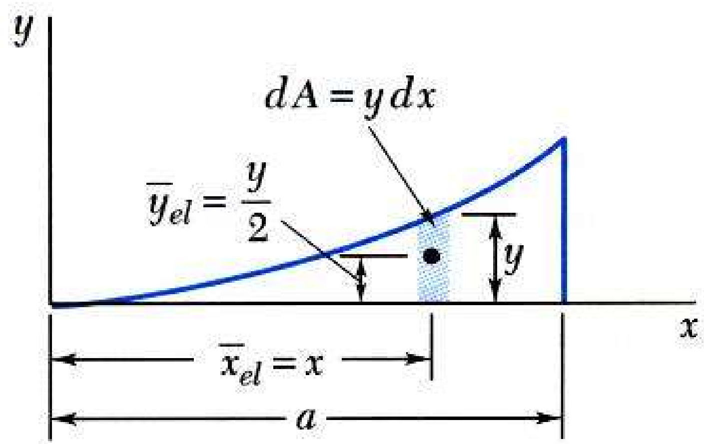
Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Sample Problem 5.4





SOLUTION:

- Determine the constant k.

$$y = k\,x^2$$

$$b = k\,a^2 \quad \Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

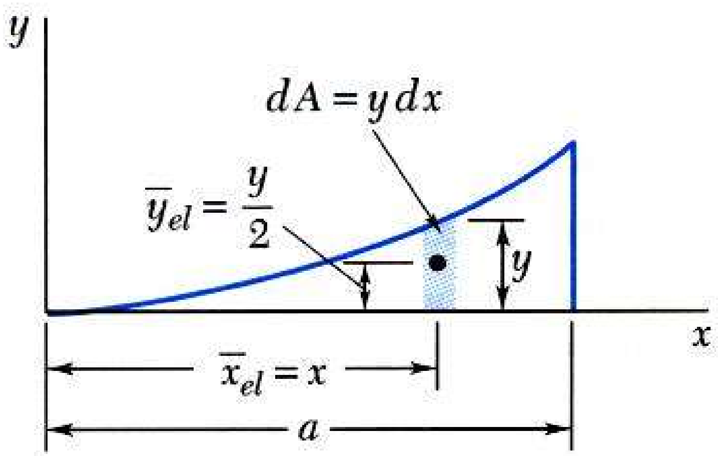
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem 5.4



- Using vertical strips, perform a single integration to find the first moments.

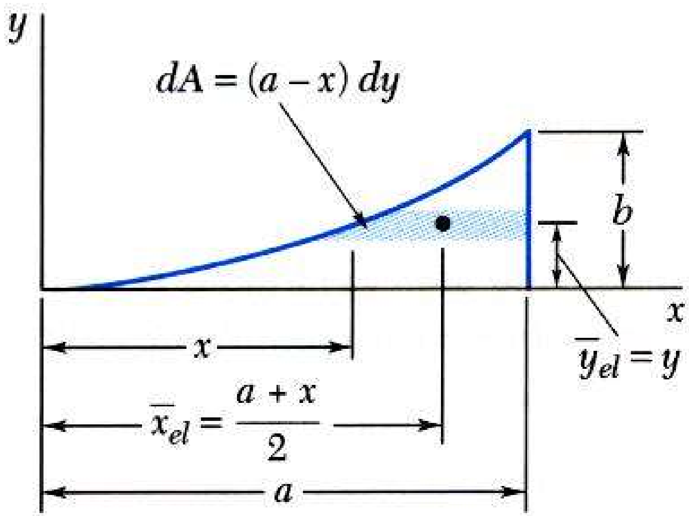
$$Q_y = \int \bar{x}_{el} dA = \int xy dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2} y dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Sample Problem 5.4



- Or, using horizontal strips, perform a single integration to find the first moments.

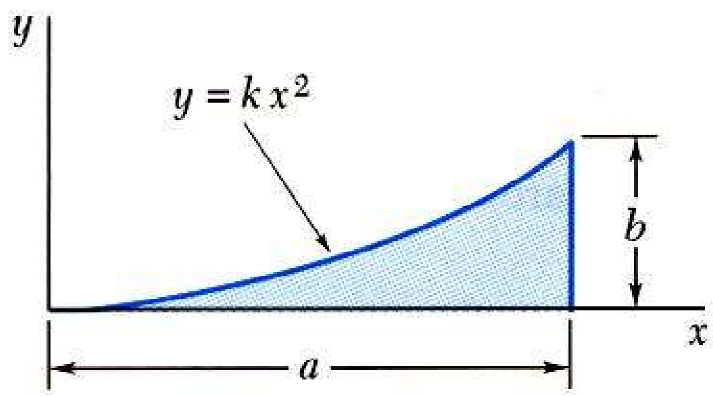
$$Q_y = \int \bar{x}_{el} dA = \int \frac{a+x}{2}(a-x)dy = \int_0^b \frac{a^2 - x^2}{2} dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y \right) dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int y(a-x)dy = \int y\left(a - \frac{a}{b^{1/2}} y^{1/2} \right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$

Sample Problem 5.4



- Evaluate the centroid coordinates.

$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4}$$

$$\boxed{\bar{x} = \frac{3}{4}a}$$

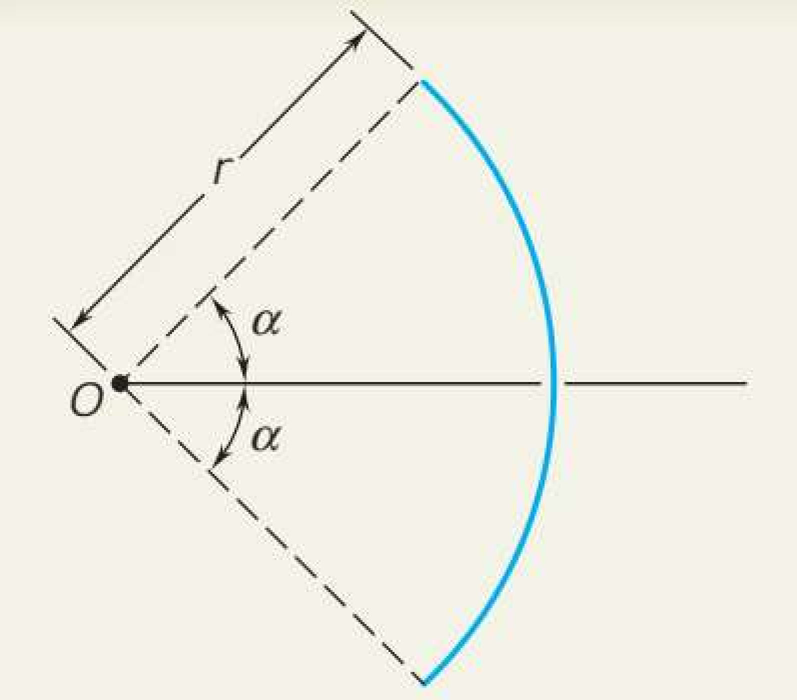
$$\bar{y}A = Q_x$$

$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10}$$

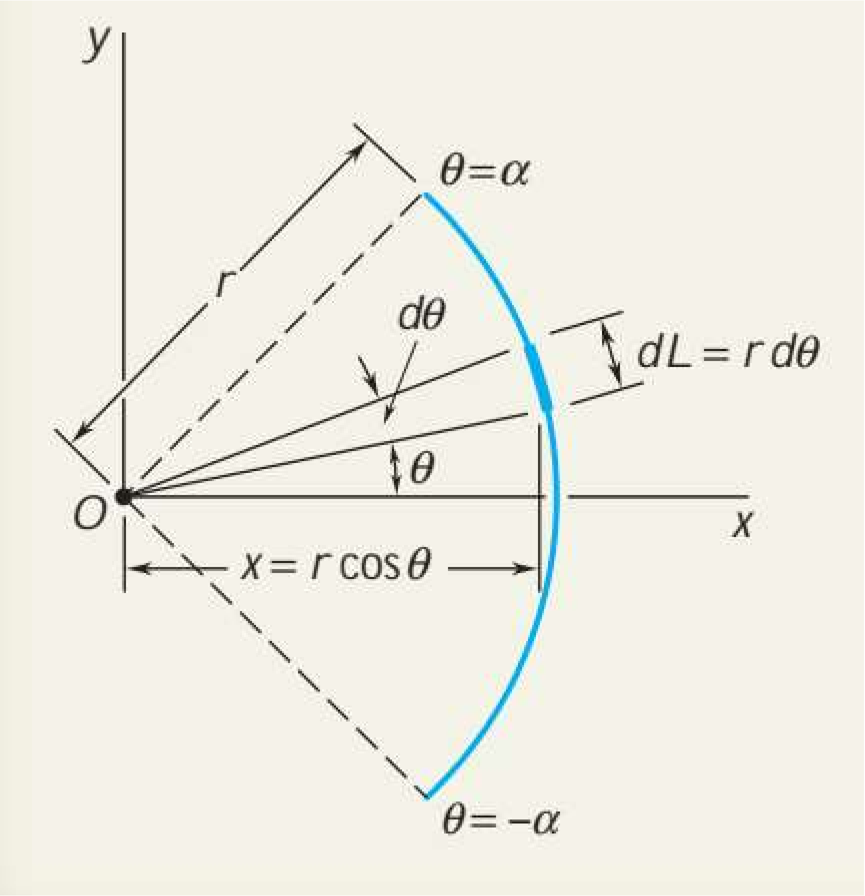
$$\boxed{\bar{y} = \frac{3}{10}b}$$

Sample Problem 5.5

Determine the location of the centroid of the arc of circle shown.



Sample Problem 5.5



Sample Problem 5.5

Since the arc is symmetrical with respect to the x axis, $\bar{y} = 0$. A differential element is chosen as shown, and the length of the arc is determined by integration.

$$L = \int dL = \int_{-\mathrm{a}}^{\mathrm{a}} r\,d\mathrm{u} = r\int_{-\mathrm{a}}^{\mathrm{a}} d\mathrm{u} = 2r\mathrm{a}$$

The first moment of the arc with respect to the y axis is

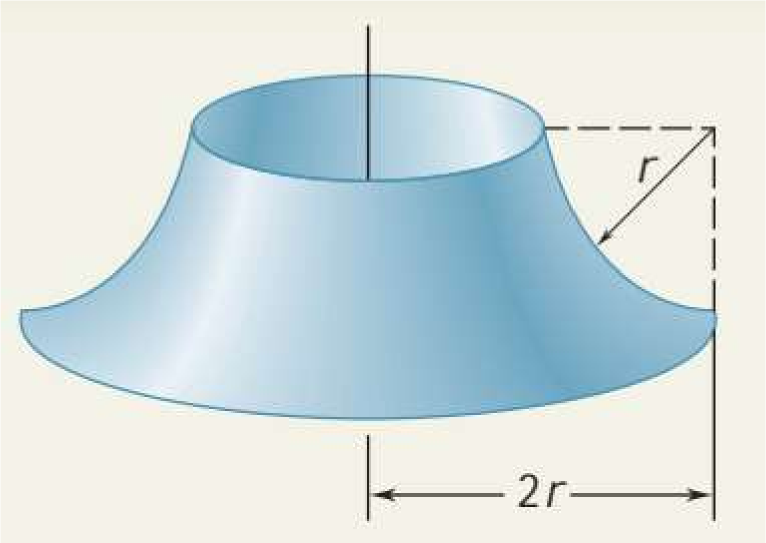
$$Q_y = \int x\,dL = \int_{-\mathrm{a}}^{\mathrm{a}} (r\cos\mathrm{u})(r\,d\mathrm{u}) = r^2\int_{-\mathrm{a}}^{\mathrm{a}} \cos\mathrm{u}\,d\mathrm{u}$$
$$= r^2[\sin\mathrm{u}]_{-\mathrm{a}}^{\mathrm{a}} = 2r^2\sin\mathrm{a}$$

Since $Q_y = \bar{x}L$, we write

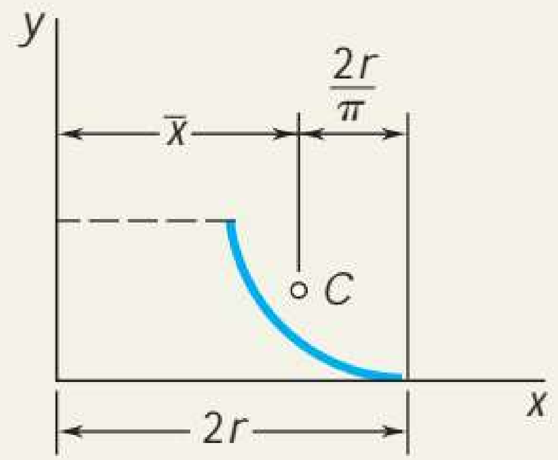
$$\bar{x}(2r\mathrm{a}) = 2r^2\sin\mathrm{a} \qquad \bar{x} = \frac{r\sin\mathrm{a}}{\mathrm{a}}$$

Sample Problem 5.6

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.



Sample Problem 5.6



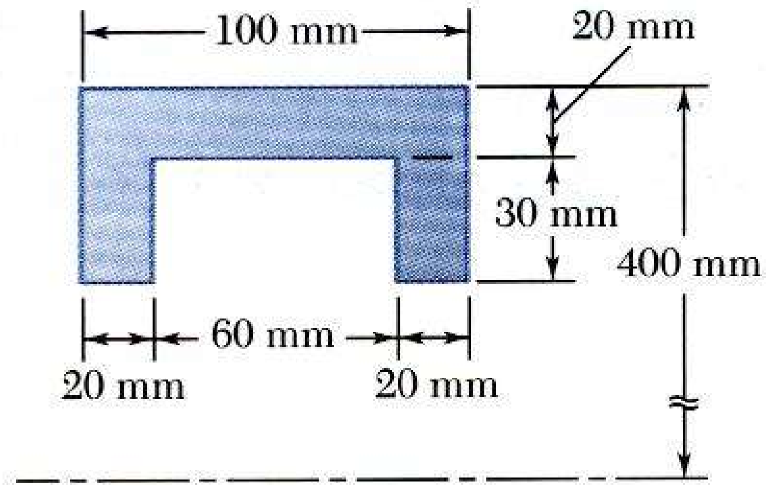
According to Theorem I of Pappus-Guldinus, the area generated is equal to the product of the length of the arc and the distance traveled by its centroid. Referring to Fig. 5.8B, we have

$$\bar{x} = 2r - \frac{2r}{\pi} = 2r\left(1 - \frac{1}{\pi}\right)$$

$$A = 2\pi\bar{x}L = 2\pi\left[2r\left(1 - \frac{1}{\pi}\right)\right]\left(\frac{\pi r}{2}\right)$$

$$A = 2\pi r^2(\pi - 1)$$

Sample Problem 5.7



The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85\times10^3\ \mathrm{kg/m^3}$ determine the mass and weight of the rim.

SOLUTION:

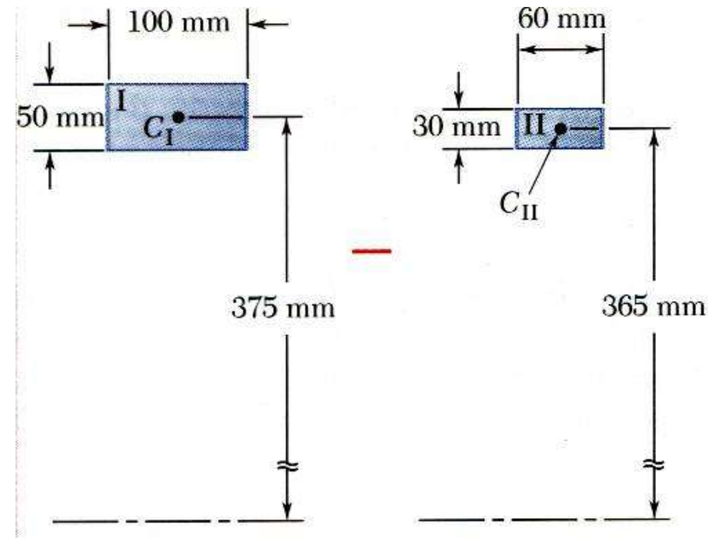
- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.
- Multiply by density and acceleration to get the mass and acceleration.

Sample Problem 5.7

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.

- Multiply by density and acceleration to get the mass and acceleration.



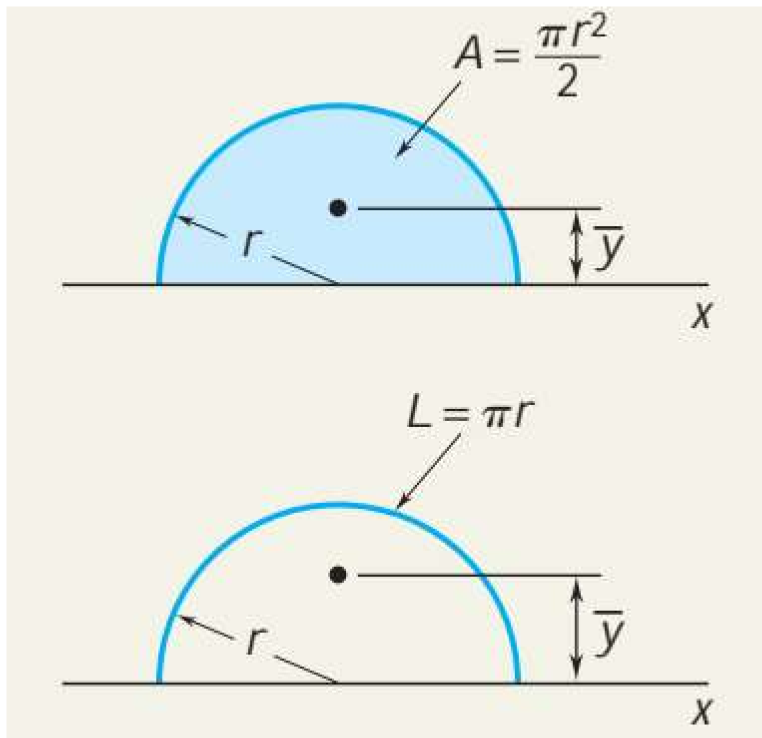
	Area, mm²	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm³
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

$$m = \rho V = \left(7.85\times10^3\,\mathrm{kg/m^3}\right)\left(7.65\times10^6\,\mathrm{mm^3}\right)\left(10^{-9}\,\mathrm{m^3/mm^3}\right) \qquad \boxed{m = 60.0\,\mathrm{kg}}$$

$$W = mg = (60.0\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right) \qquad \boxed{W = 589\,\mathrm{N}}$$

Sample Problem 5.8

Using the theorems of Pappus-Guldinus, determine (*a*) the centroid of a semicircular area, (*b*) the centroid of a semicircular arc. We recall that the volume and the surface area of a sphere are $\frac{4}{3}\pi r^3$ and $4\pi r^2$, respectively.



Sample Problem 5.8

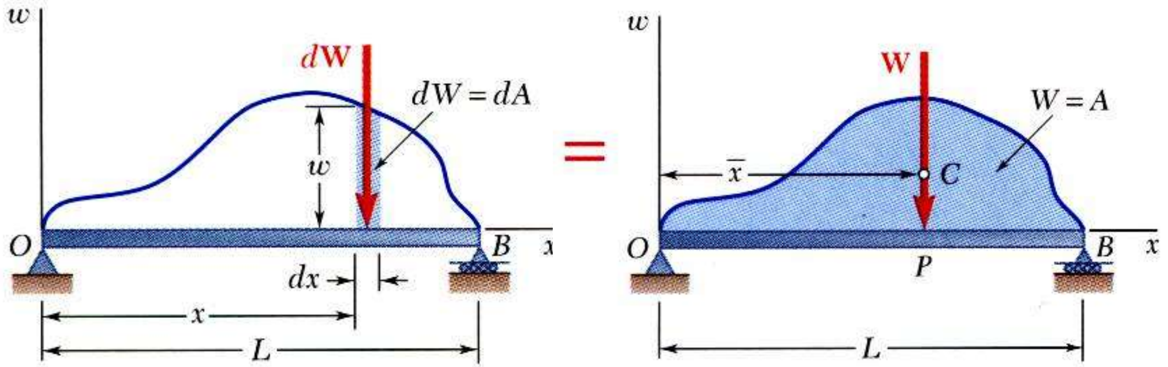
The volume of a sphere is equal to the product of the area of a semicircle and the distance traveled by the centroid of the semicircle in one revolution about the x axis.

$$V = 2\mathrm{p}\bar{y}A \qquad \tfrac{4}{3}\mathrm{p}r^3 = 2\mathrm{p}\bar{y}(\tfrac{1}{2}\mathrm{p}r^2) \qquad \bar{y} = \frac{4r}{3\mathrm{p}}$$

Likewise, the area of a sphere is equal to the product of the length of the generating semicircle and the distance traveled by its centroid in one revolution.

$$A = 2\mathrm{p}\bar{y}L \qquad 4\mathrm{p}r^2 = 2\mathrm{p}\bar{y}(\mathrm{p}r) \qquad \bar{y} = \frac{2r}{\mathrm{p}}$$

Distributed Loads on Beams



$$W = \int_0^L w\,dx = \int dA = A$$

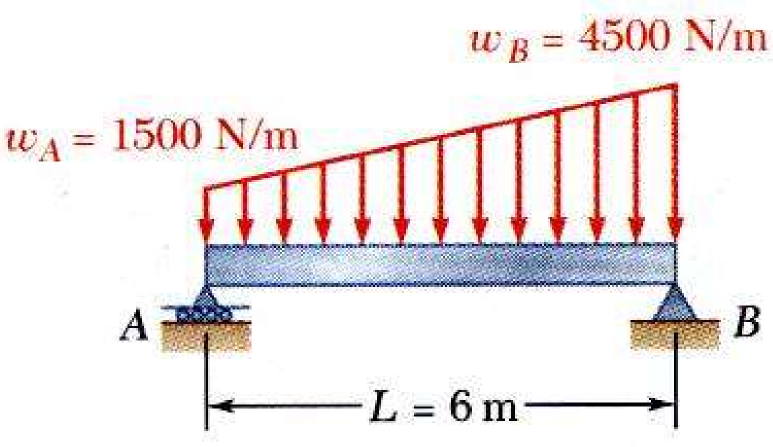
- A distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.

$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

Sample Problem 5.9

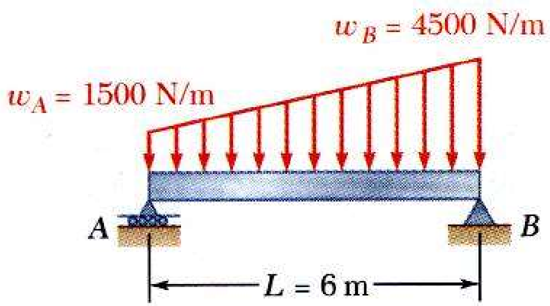


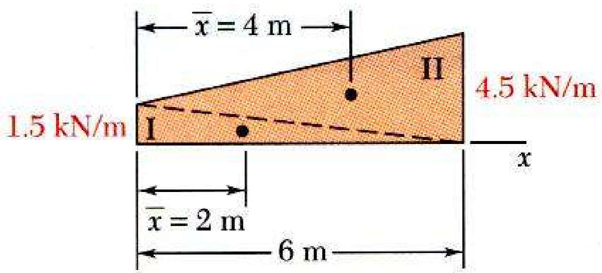
A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

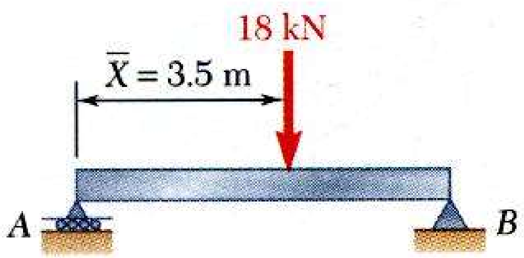
SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

Sample Problem 5.9







SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.

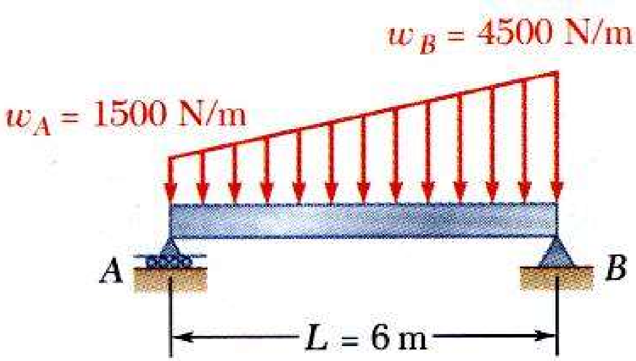
$$\boxed{F = 18.0\,\text{kN}}$$

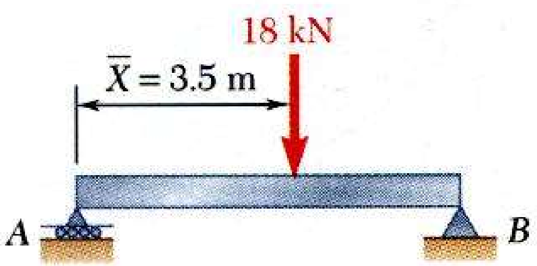
- The line of action of the concentrated load passes through the centroid of the area under the curve.

$$\bar{X} = \frac{63\,\text{kN}\cdot\text{m}}{18\,\text{kN}} \qquad \boxed{\bar{X} = 3.5\,\text{m}}$$

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma\bar{x}A = 63$

Sample Problem 5.9





- Determine the support reactions by summing moments about the beam ends.

$$\sum M_A = 0: \quad B_y(6\,\mathrm{m}) - (18\,\mathrm{kN})(3.5\,\mathrm{m}) = 0$$

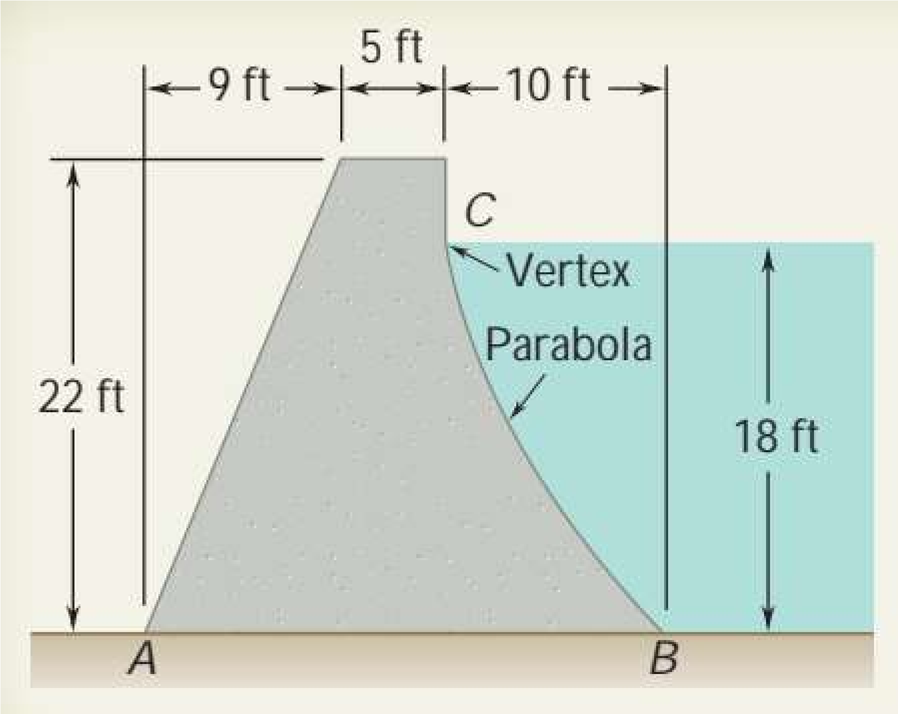
$$\boxed{B_y = 10.5\,\mathrm{kN}}$$

$$\sum M_B = 0: \quad -A_y(6\,\mathrm{m}) + (18\,\mathrm{kN})(6\,\mathrm{m} - 3.5\,\mathrm{m}) = 0$$

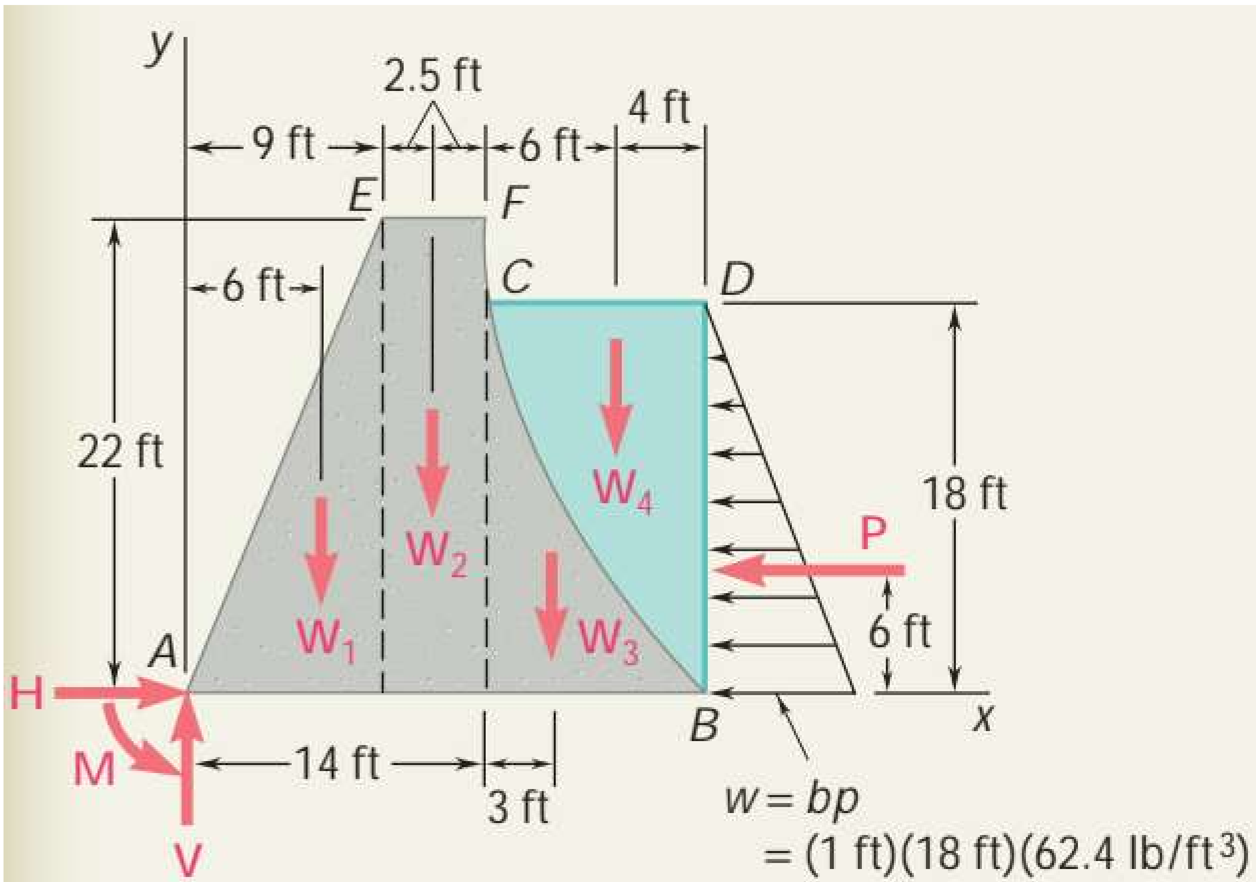
$$\boxed{A_y = 7.5\,\mathrm{kN}}$$

Sample Problem 5.10

The cross section of a concrete dam is as shown. Consider a 1-ft-thick section of the dam, and determine (*a*) the resultant of the reaction forces exerted by the ground on the base *AB* of the dam, (*b*) the resultant of the pressure forces exerted by the water on the face *BC* of the dam. The specific weights of concrete and water are 150 lb/ft^3 and 62.4 lb/ft^3, respectively.



Sample Problem 5.10



Sample Problem 5.10

a. Ground Reaction. We choose as a free body the 1-ft-thick section *AEFCDB* of the dam and water. The reaction forces exerted by the ground on the base *AB* are represented by an equivalent force-couple system at *A*. Other forces acting on the free body are the weight of the dam, represented by the weights of its components $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_3$; the weight of the water $\mathbf{W}_4$; and the resultant $\mathbf{P}$ of the pressure forces exerted on section *BD* by the water to the right of section *BD*. We have

$$W_1 = \tfrac{1}{2}(9\text{ ft})(22\text{ ft})(1\text{ ft})(150\text{ lb/ft}^3) = 14{,}850\text{ lb}$$
$$W_2 = (5\text{ ft})(22\text{ ft})(1\text{ ft})(150\text{ lb/ft}^3) = 16{,}500\text{ lb}$$
$$W_3 = \tfrac{1}{3}(10\text{ ft})(18\text{ ft})(1\text{ ft})(150\text{ lb/ft}^3) = 9000\text{ lb}$$
$$W_4 = \tfrac{2}{3}(10\text{ ft})(18\text{ ft})(1\text{ ft})(62.4\text{ lb/ft}^3) = 7488\text{ lb}$$
$$P = \tfrac{1}{2}(18\text{ ft})(1\text{ ft})(18\text{ ft})(62.4\text{ lb/ft}^3) = 10{,}109\text{ lb}$$

Sample Problem 5.10

Equilibrium Equations

$\overset{+}{y}\ \Sigma F_x = 0$: $\quad H - 10{,}109 \text{ lb} = 0$ $\qquad \mathbf{H} = 10{,}110 \text{ lb y}$ ◀

$+\mathrm{x}\ \Sigma F_y = 0$: $\quad V - 14{,}850 \text{ lb} - 16{,}500 \text{ lb} - 9000 \text{ lb} - 7488 \text{ lb} = 0$

$\mathbf{V} = 47{,}840 \text{ lbx}$ ◀

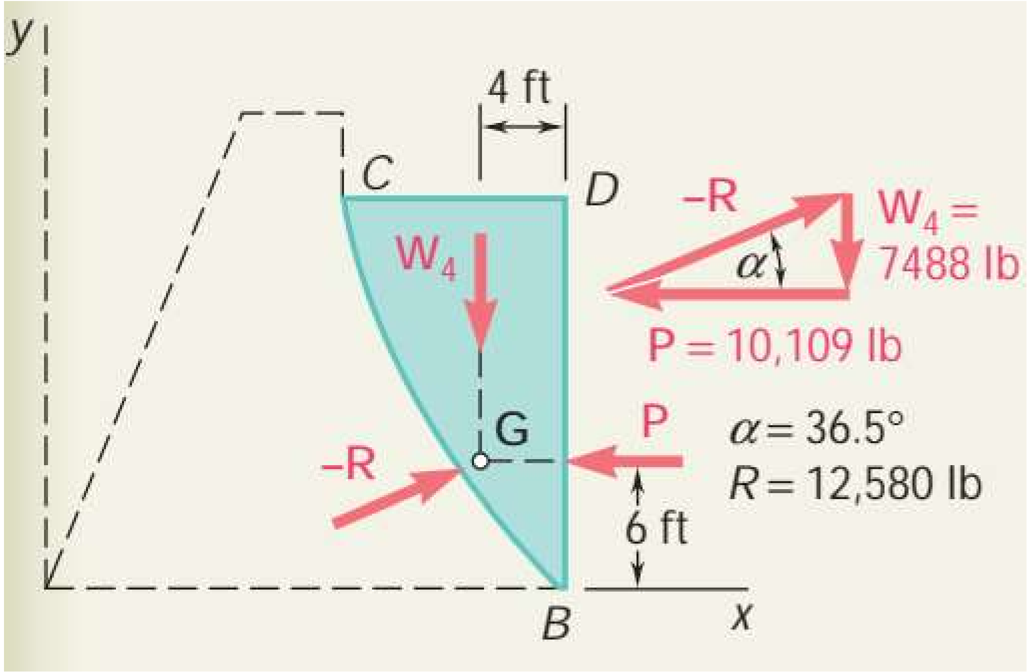
$+\mathrm{l}\ \Sigma M_A = 0$: $\quad -(14{,}850 \text{ lb})(6 \text{ ft}) - (16{,}500 \text{ lb})(11.5 \text{ ft})$
$\qquad - (9000 \text{ lb})(17 \text{ ft}) - (7488 \text{ lb})(20 \text{ ft}) + (10{,}109 \text{ lb})(6 \text{ ft}) + M = 0$

$\mathbf{M} = 520{,}960 \text{ lb} \cdot \text{ft l}$ ◀

We can replace the force-couple system obtained by a single force acting at a distance d to the right of A, where

$$d = \frac{520{,}960 \text{ lb} \cdot \text{ft}}{47{,}840 \text{ lb}} = 10.89 \text{ ft}$$

Sample Problem 5.10

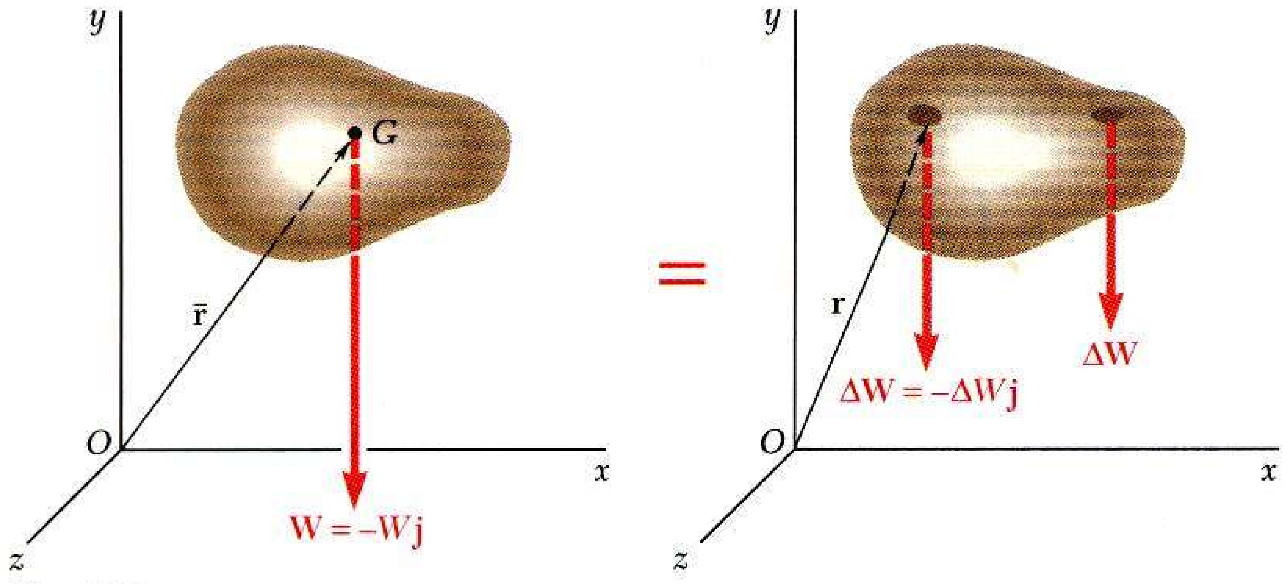


Sample Problem 5.10

b. Resultant R of Water Forces. The parabolic section of water *BCD* is chosen as a free body. The forces involved are the resultant $-\mathbf{R}$ of the forces exerted by the dam on the water, the weight $\mathbf{W}_4$, and the force $\mathbf{P}$. Since these forces must be concurrent, $-\mathbf{R}$ passes through the point of intersection *G* of $\mathbf{W}_4$ and $\mathbf{P}$. A force triangle is drawn from which the magnitude and direction of $-\mathbf{R}$ are determined. The resultant $\mathbf{R}$ of the forces exerted by the water on the face *BC* is equal and opposite:

$$\mathbf{R} = 12{,}580 \text{ lb} \measuredangle 36.5°$$

Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity G

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times\left(-W\vec{j}\right) = \sum\left[\vec{r}\times\left(-\Delta W\vec{j}\right)\right]$$
$$\vec{r}_G W \times\left(-\vec{j}\right) = \left(\sum \vec{r}\Delta W\right)\times\left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\, dW$$

- Results are independent of body orientation,

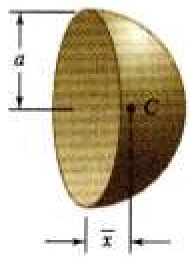
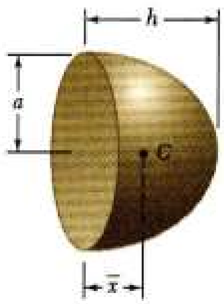
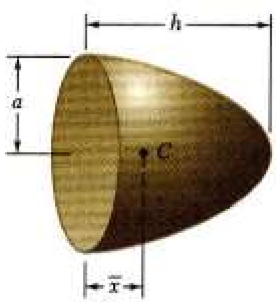
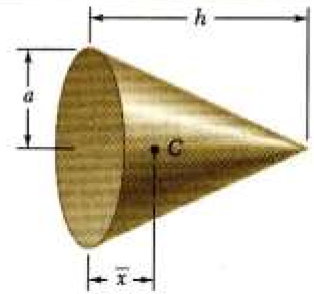
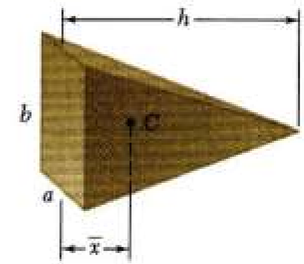
$$\bar{x}W = \int x\,dW \quad \bar{y}W = \int y\,dW \quad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

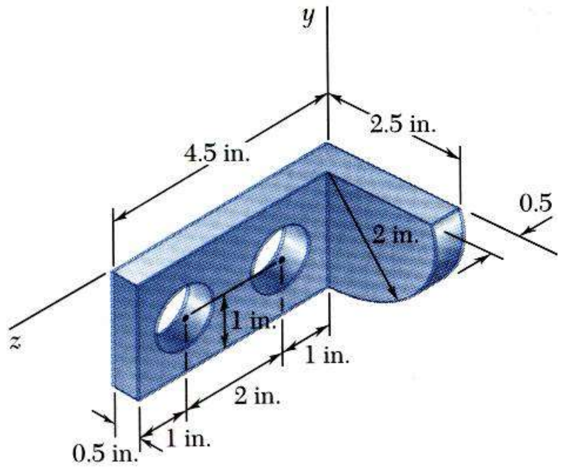
$$W = \gamma V \text{ and } dW = \gamma\, dV$$

$$\bar{x}V = \int x\,dV \quad \bar{y}V = \int y\,dV \quad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Composite 3D Bodies

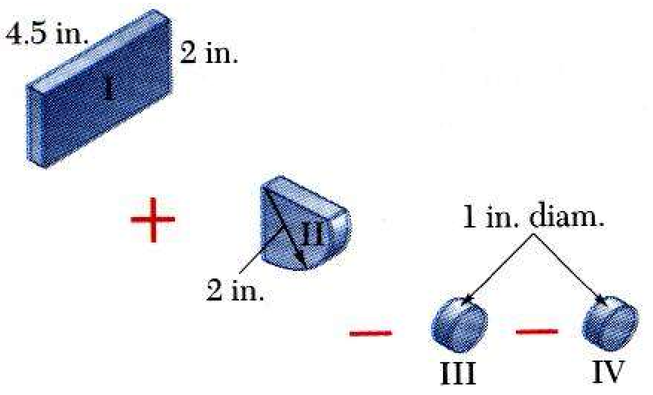


- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

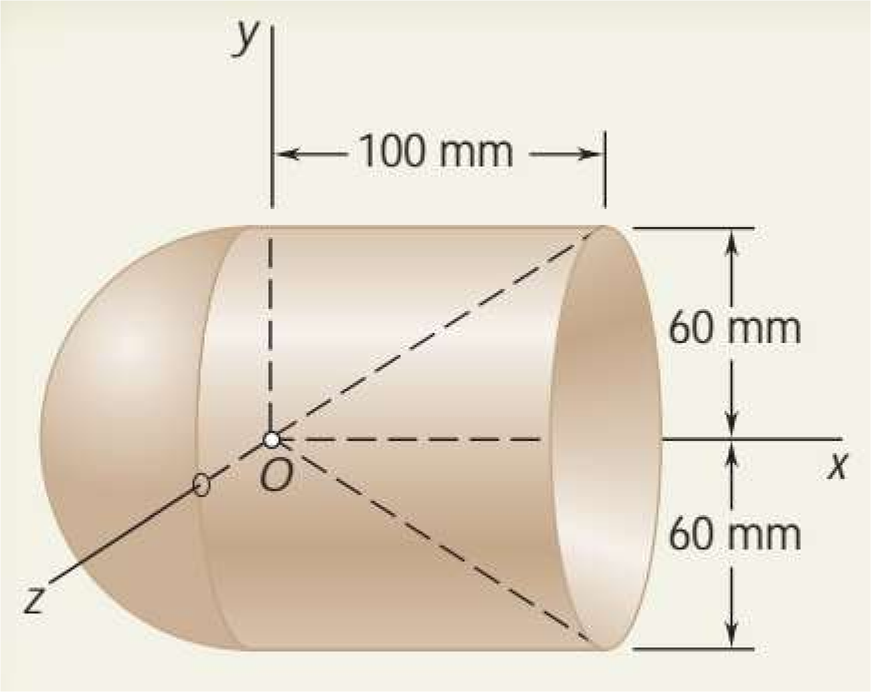
- For homogeneous bodies,

$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$



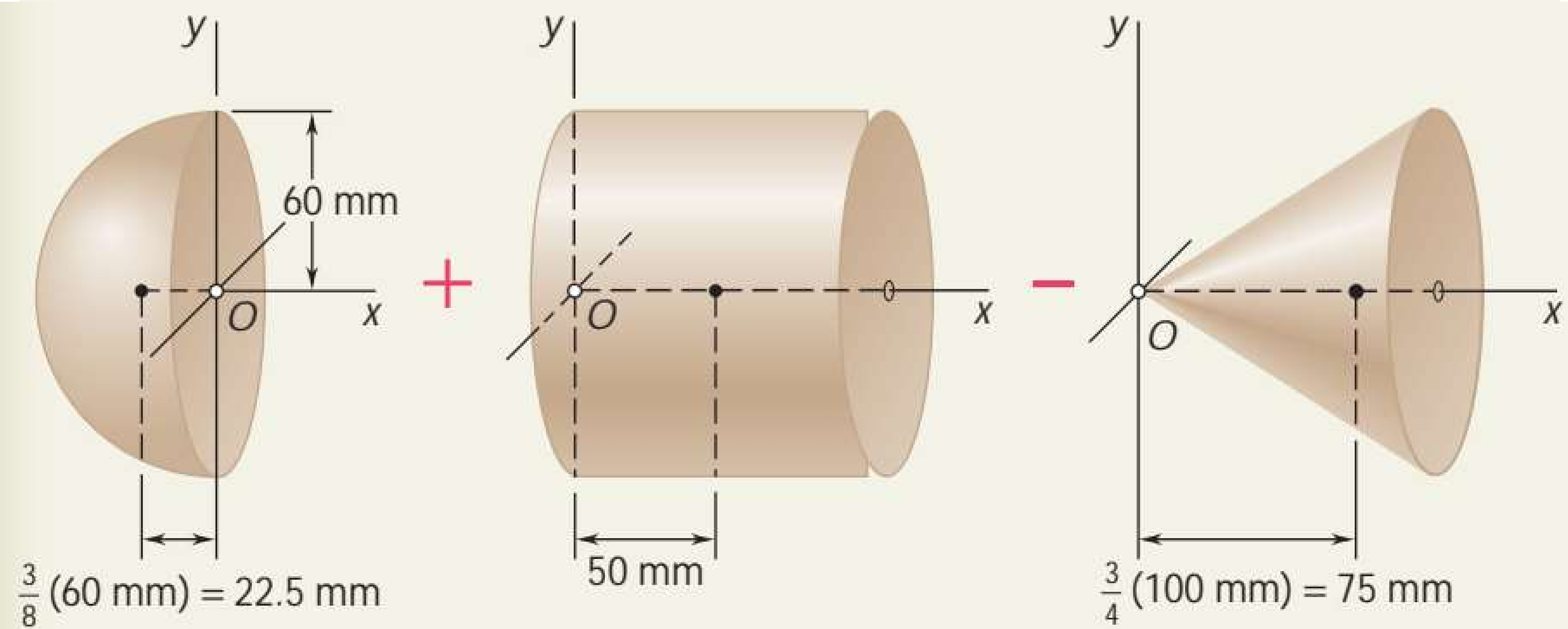
Sample Problem 5.11

Determine the location of the center of gravity of the homogeneous body of revolution shown, which was obtained by joining a hemisphere and a cylinder and carving out a cone.



Sample Problem 5.11

Because of symmetry, the center of gravity lies on the x axis. As shown in the figure below, the body can be obtained by adding a hemisphere to a cylinder and then subtracting a cone. The volume and the abscissa of the centroid of each of these components are obtained from Fig. 5.21 and are entered in the table below. The total volume of the body and the first moment of its volume with respect to the yz plane are then determined.



Sample Problem 5.11

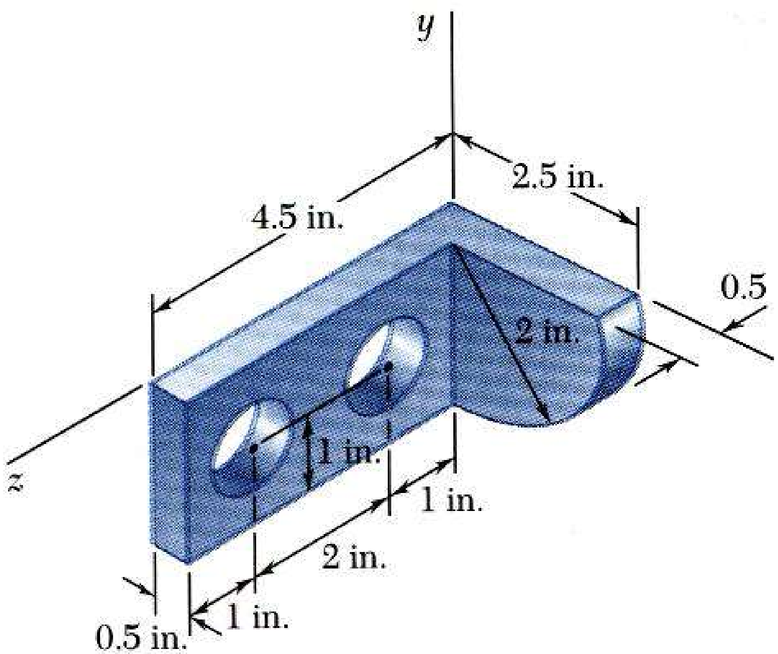
Component	Volume, mm^3	$\bar{x}$, mm	$\bar{x}V$, mm^4
Hemisphere	$\frac{1}{2}\frac{4p}{3}(60)^3 = 0.4524 \times 10^6$	-22.5	-10.18×10^6
Cylinder	$p(60)^2(100) = 1.1310 \times 10^6$	$+50$	$+56.55 \times 10^6$
Cone	$-\frac{p}{3}(60)^2(100) = -0.3770 \times 10^6$	$+75$	-28.28×10^6
	$\Sigma V = 1.206 \times 10^6$		$\Sigma \bar{x}V = +18.09 \times 10^6$

Thus,

$$\bar{X}\Sigma V = \Sigma \bar{x}V: \qquad \bar{X}(1.206 \times 10^6 \text{ mm}^3) = 18.09 \times 10^6 \text{ mm}^4$$

$$\bar{X} = 15 \text{ mm}$$

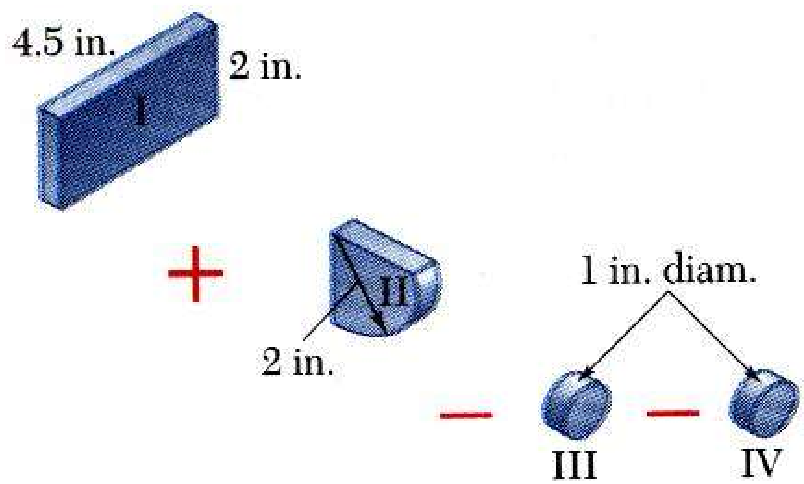
Sample Problem 5.12



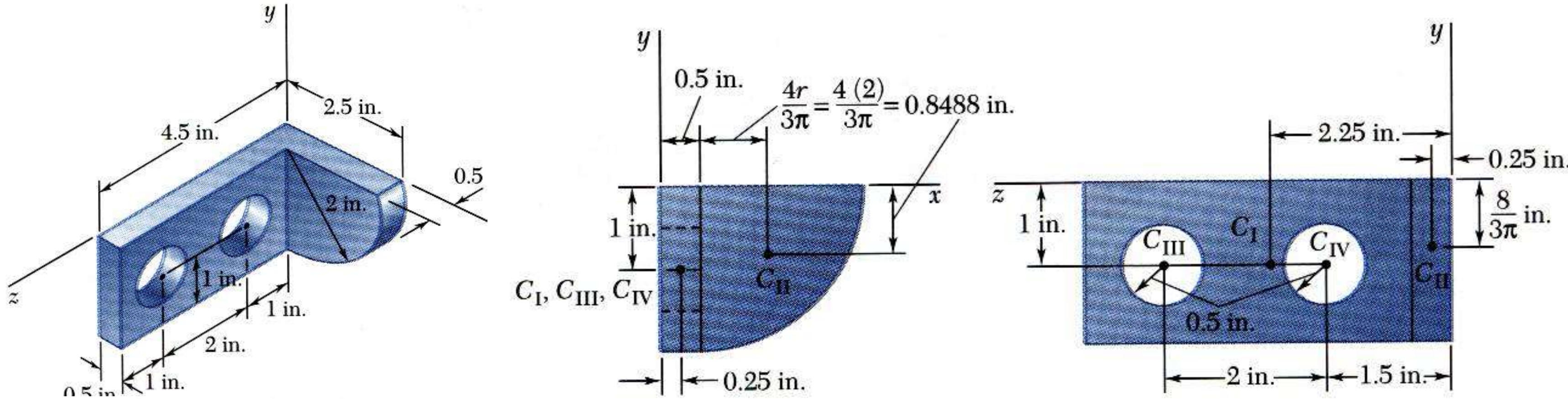
Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.

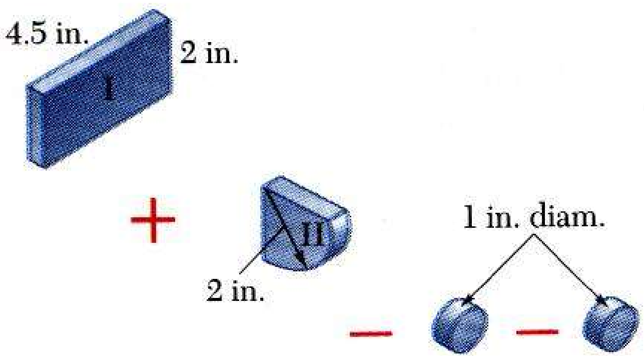
SOLUTION:

- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.



Sample Problem 5.12

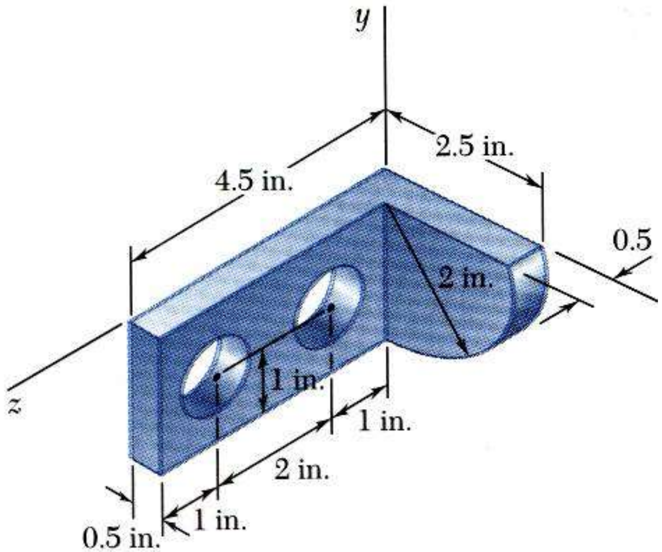




	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

Sample Problem 5.12

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$



$$\bar{X} = \sum \bar{x}V / \sum V = \left(3.08\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{X} = 0.577\,\text{in.}}$$

$$\bar{Y} = \sum \bar{y}V / \sum V = \left(-5.047\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

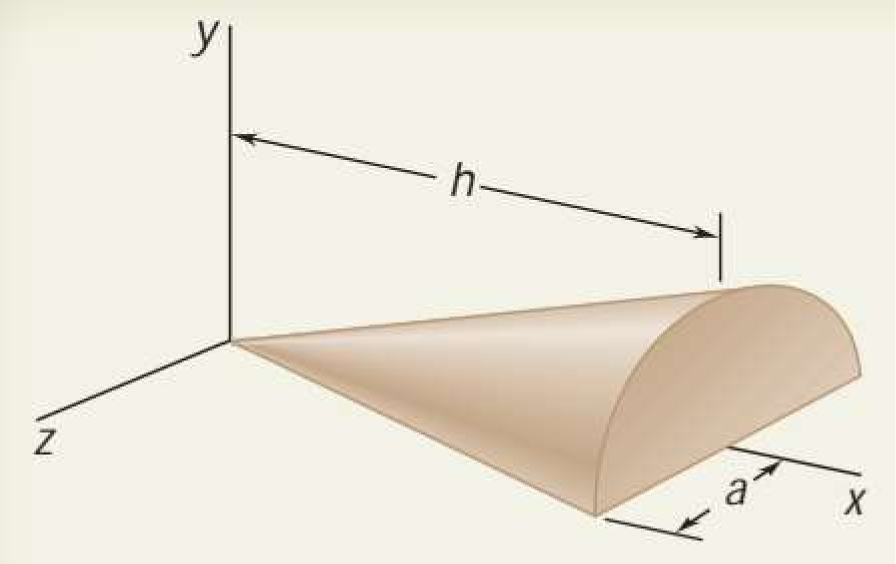
$$\boxed{\bar{Y} = 0.577\,\text{in.}}$$

$$\bar{Z} = \sum \bar{z}V / \sum V = \left(1.618\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

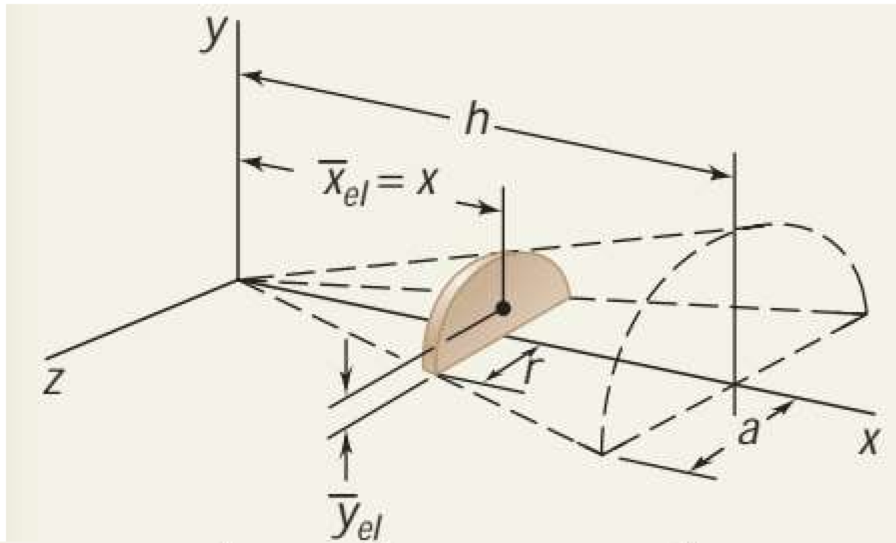
$$\boxed{\bar{Z} = 0.577\,\text{in.}}$$

Sample Problem 5.13

Determine the location of the centroid of the half right circular cone shown.



Sample Problem 5.13



Since the xy plane is a plane of symmetry, the centroid lies in this plane and $\bar{z} = 0$. A slab of thickness dx is chosen as a differential element. The volume of this element is

$$dV = \tfrac{1}{2}\mathrm{p}r^2\,dx$$

The coordinates $\bar{x}_{el}$ and $\bar{y}_{el}$ of the centroid of the element are obtained from Fig. 5.8 (semicircular area).

$$\bar{x}_{el} = x \qquad\qquad \bar{y}_{el} = \frac{4r}{3\mathrm{p}}$$

Sample Problem 5.13

We observe that r is proportional to x and write

$$\frac{r}{x} = \frac{a}{h} \qquad\qquad r = \frac{a}{h}x$$

The volume of the body is

$$V = \int dV = \int_0^h \tfrac{1}{2}\mathrm{p}r^2\,dx = \int_0^h \tfrac{1}{2}\mathrm{p}\left(\frac{a}{h}x\right)^2 dx = \frac{\mathrm{p}a^2h}{6}$$

The moment of the differential element with respect to the yz plane is $\bar{x}_{el}\,dV$; the total moment of the body with respect to this plane is

$$\int \bar{x}_{el}\,dV = \int_0^h x(\tfrac{1}{2}\mathrm{p}r^2)\,dx = \int_0^h x(\tfrac{1}{2}\mathrm{p})\left(\frac{a}{h}x\right)^2 dx = \frac{\mathrm{p}a^2h^2}{8}$$

Thus,

$$\bar{x}V = \int \bar{x}_{el}\,dV \qquad\qquad \bar{x}\frac{\mathrm{p}a^2h}{6} = \frac{\mathrm{p}a^2h^2}{8} \qquad\qquad \bar{x} = \tfrac{3}{4}h$$

Sample Problem 5.13

Likewise, the moment of the differential element with respect to the zx plane is $\bar{y}_{el}\, dV$; the total moment is

$$\int \bar{y}_{el}\, dV = \int_0^h \frac{4r}{3\pi}\left(\tfrac{1}{2}\pi r^2\right) dx = \frac{2}{3}\int_0^h \left(\frac{a}{h}x\right)^3 dx = \frac{a^3 h}{6}$$

Thus,

$$\bar{y}V = \int \bar{y}_{el}\, dV \qquad \bar{y}\,\frac{\pi a^2 h}{6} = \frac{a^3 h}{6} \qquad \bar{y} = \frac{a}{\pi}$$

CHAPTER

6

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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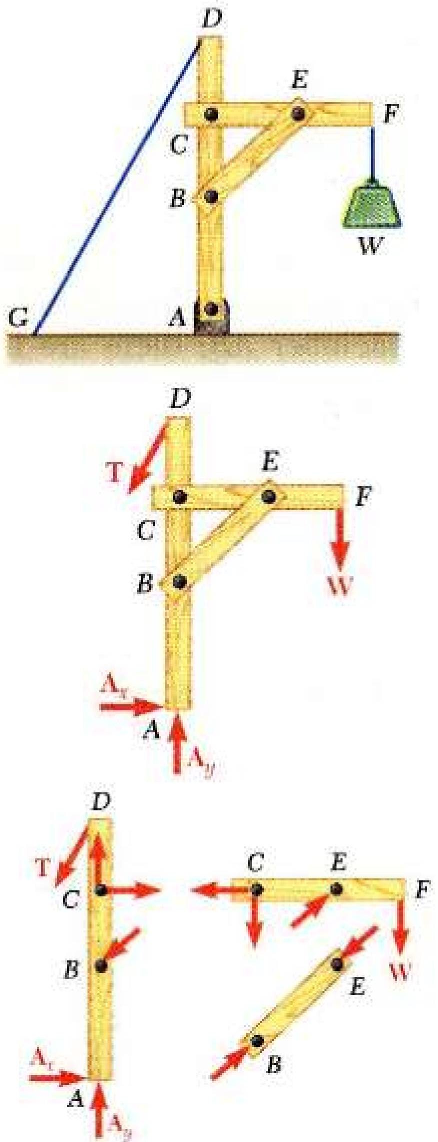
Analysis of Structures

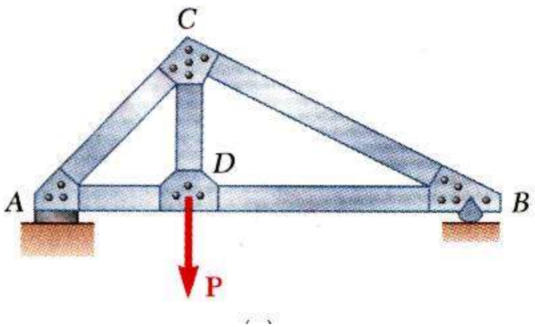
Contents

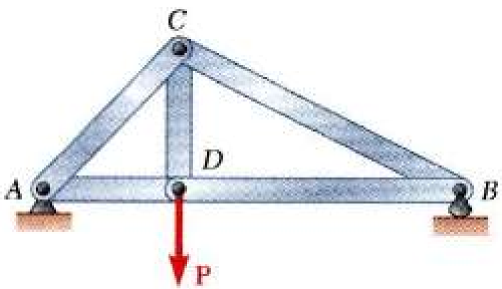
Introduction

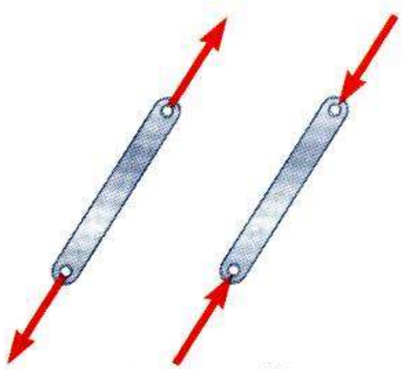


- For the equilibrium of structures made of several connected parts, the *internal forces* as well the *external forces* are considered.
- In the interaction between connected parts, Newton's 3rd Law states that the *forces of action and reaction* between bodies in contact have the same magnitude, same line of action, and opposite sense.
- Three categories of engineering structures are considered:
  - *a) Frames*: contain at least one one multi-force member, i.e., member acted upon by 3 or more forces.
  - *b) Trusses*: formed from *two-force members*, i.e., straight members with end point connections
  - *c) Machines*: structures containing moving parts designed to transmit and modify forces.

Definition of a Truss







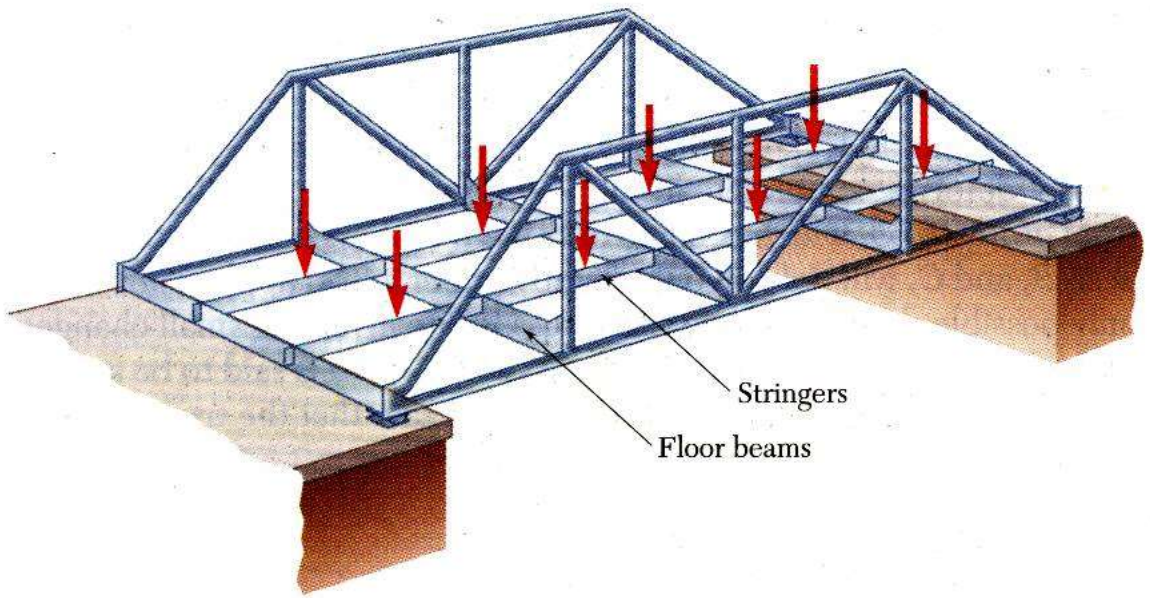
- A truss consists of straight members connected at joints. No member is continuous through a joint.

- Most structures are made of several trusses joined together to form a space framework. Each truss carries those loads which act in its plane and may be treated as a two-dimensional structure.

- Bolted or welded connections are assumed to be pinned together. Forces acting at the member ends reduce to a single force and no couple. Only *two-force members* are considered.

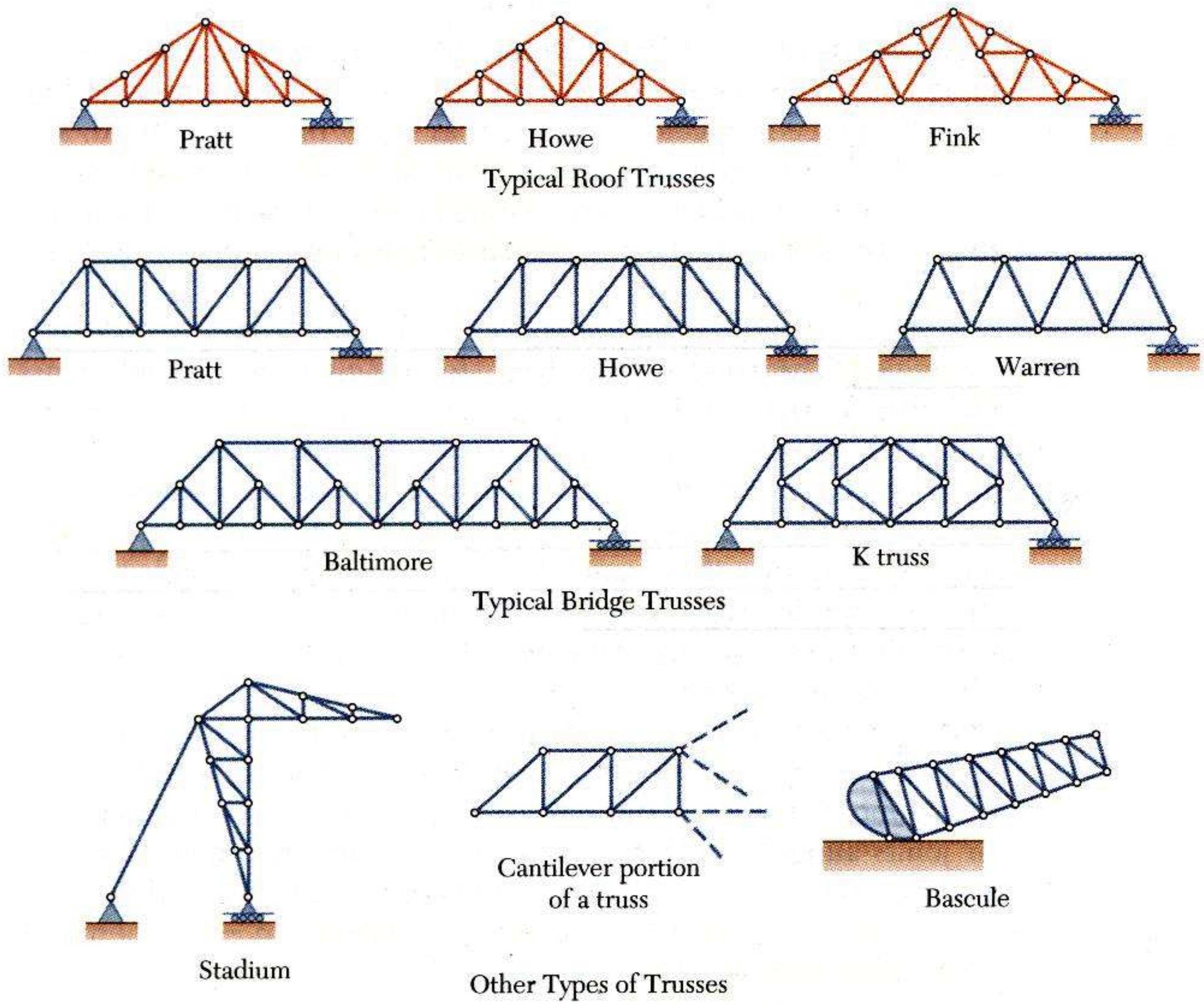
- When forces tend to pull the member apart, it is in *tension*. When the forces tend to compress the member, it is in *compression*.

Definition of a Truss



Members of a truss are slender and not capable of supporting large lateral loads. Loads must be applied at the joints.

Definition of a Truss



Examples of Trusses

Roof trusses – Safeco Field in Seattle

Examples of Trusses

Photo 6.1 - Pin-jointed connection of the approach span to the San Francisco-Oakland Bay Bridge

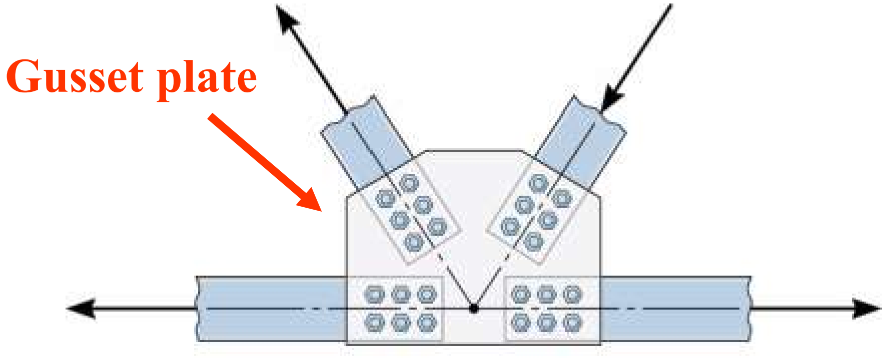
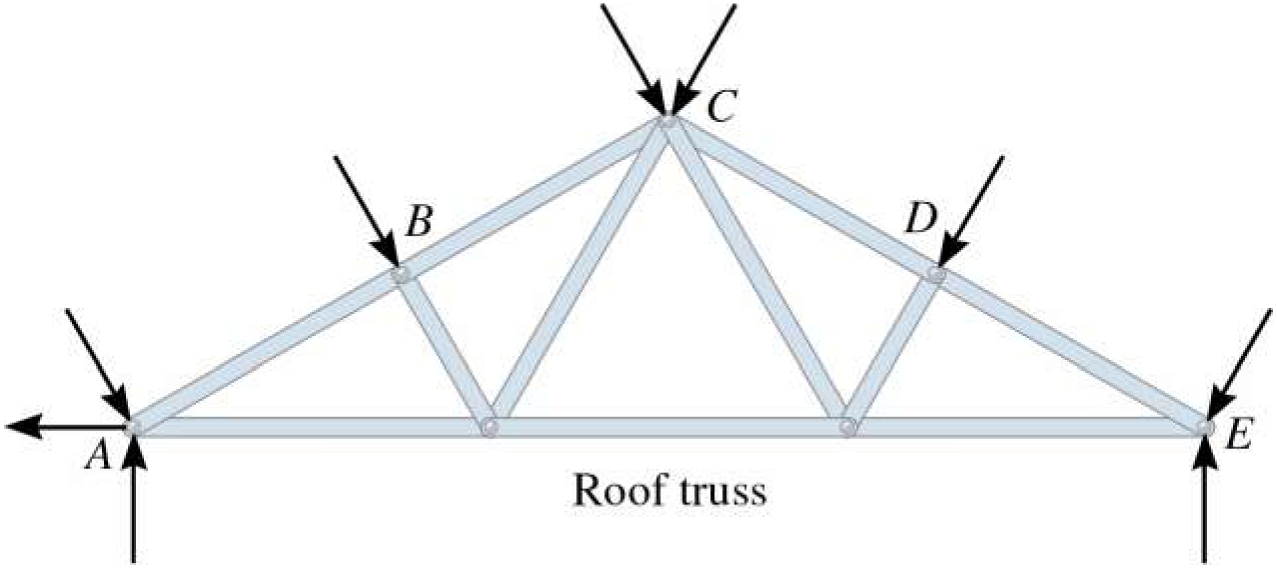
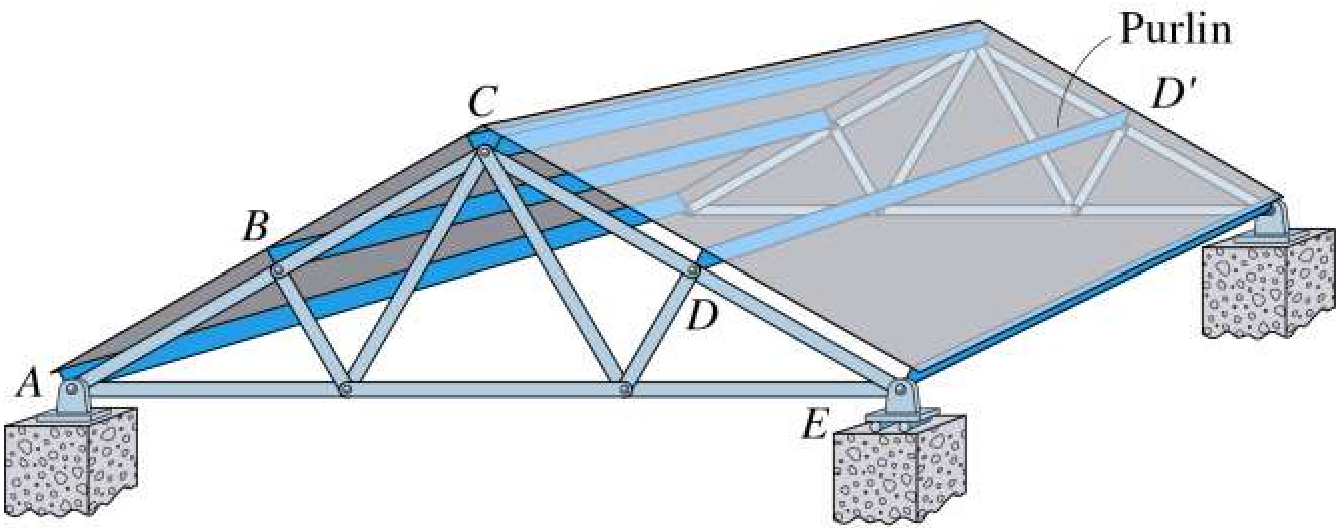


Figure 06.01(a)

Joints are often bolted, riveted, or welded. Gusset plates are also often included to tie the members together. However, the members are designed to support axial loads so assuming that the joints act as if they are pinned is a good approximation.

Examples of Trusses





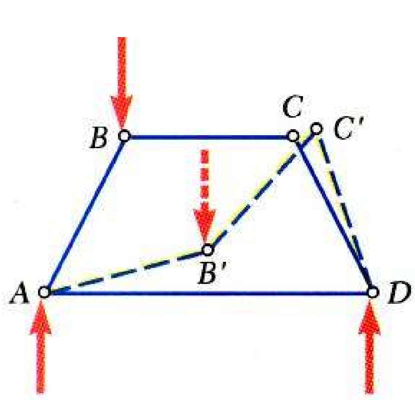
The roof truss shown is formed by two <u>planar trusses</u> connected by a series of purlins.

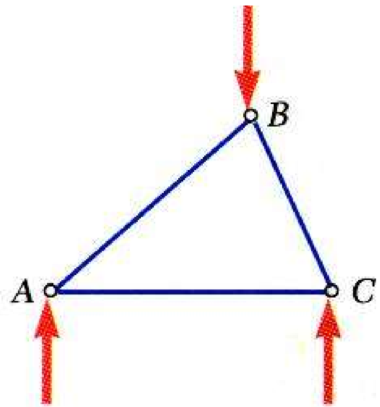
Reference: <u>Statics</u>, 9th Edition, by Hibbeler.

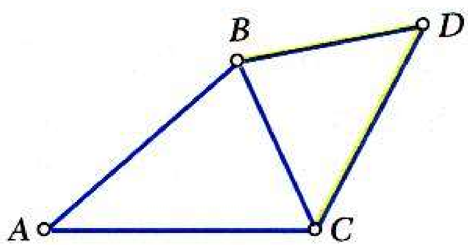
Examples of Trusses

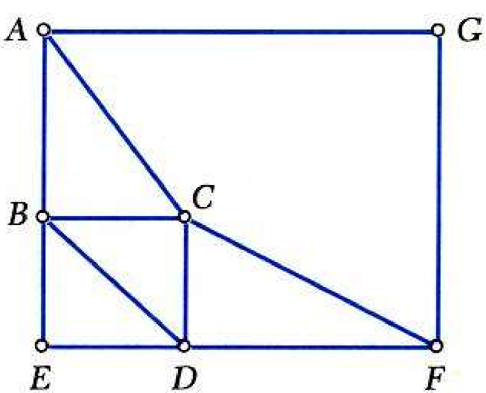
Photo 6.3 - Because roof trusses, such as those shown, require support only at their ends, it is possible to construct buildings with large unobstructed floor areas.

Simple Trusses



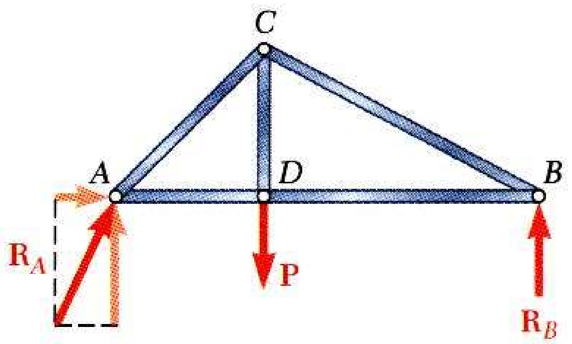


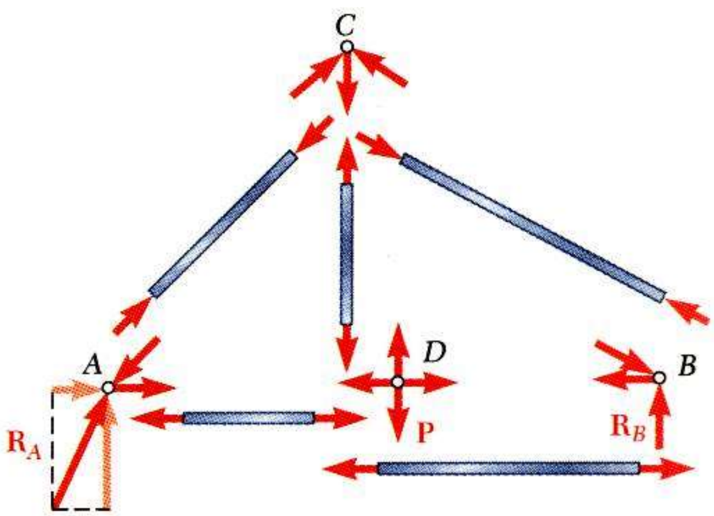




- A *rigid truss* will not collapse under the application of a load.
- A *simple truss* is constructed by successively adding two members and one connection to the basic triangular truss.
- In a simple truss, $m = 2n - 3$ where m is the total number of members and n is the number of joints.

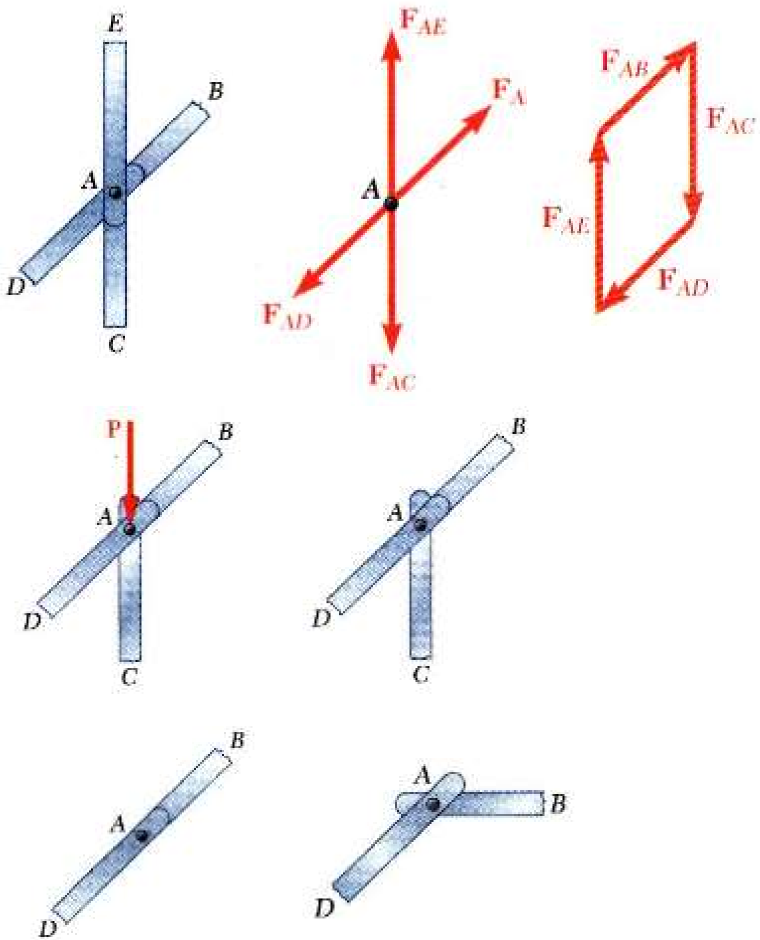
Analysis of Trusses by the Method of Joints



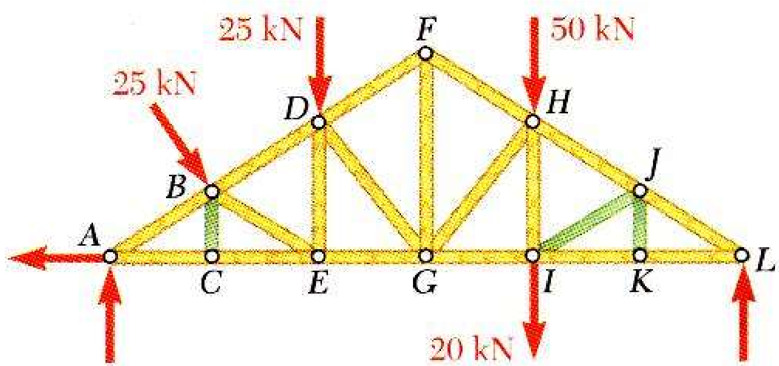


- Dismember the truss and create a freebody diagram for each member and pin.
- The two forces exerted on each member are equal, have the same line of action, and opposite sense.
- Forces exerted by a member on the pins or joints at its ends are directed along the member and equal and opposite.
- Conditions of equilibrium on the pins provide $2n$ equations for $2n$ unknowns. For a simple truss, $2n = m + 3$. May solve for m member forces and 3 reaction forces at the supports.
- Conditions for equilibrium for the entire truss provide 3 additional equations which are not independent of the pin equations.

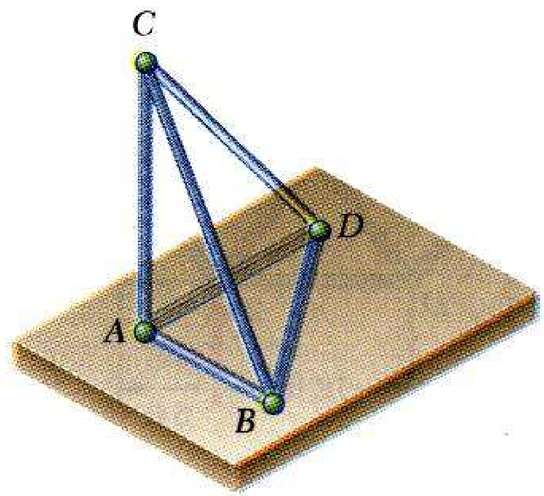
Joints Under Special Loading Conditions

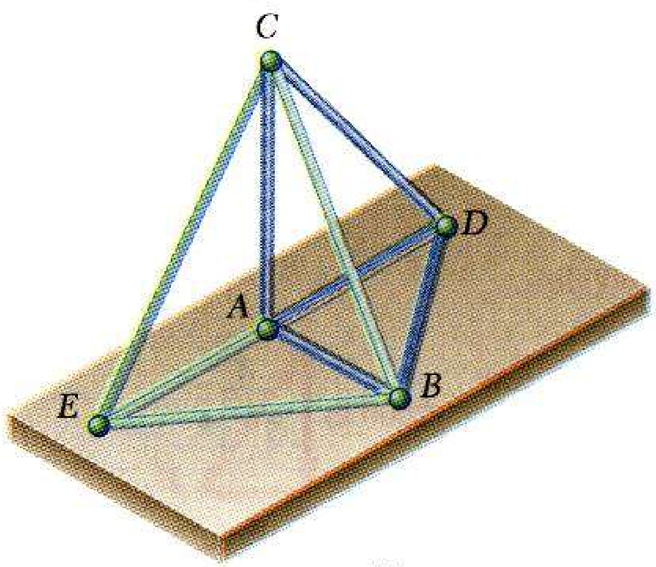


- Forces in opposite members intersecting in two straight lines at a joint are equal.
- The forces in two opposite members are equal when a load is aligned with a third member. The third member force is equal to the load (including zero load).
- The forces in two members connected at a joint are equal if the members are aligned and zero otherwise.
- Recognition of joints under special loading conditions simplifies a truss analysis.



Space Trusses



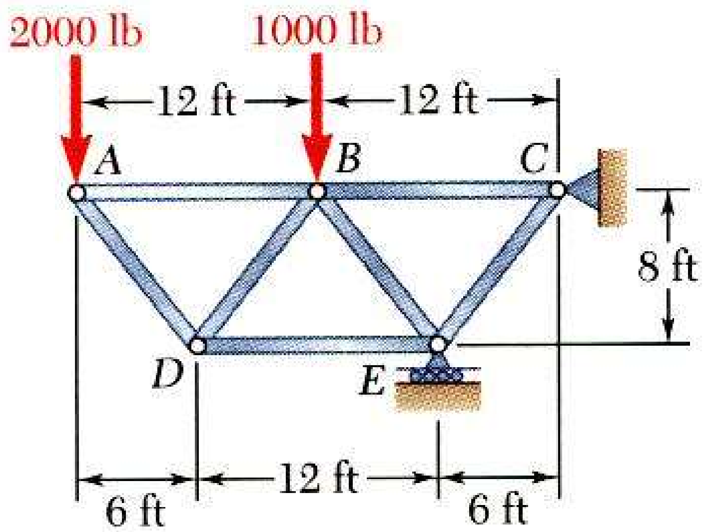


- An *elementary space truss* consists of 6 members connected at 4 joints to form a tetrahedron.
- A *simple space truss* is formed and can be extended when 3 new members and 1 joint are added at the same time.
- In a simple space truss, $m = 3n - 6$ where m is the number of members and n is the number of joints.
- Conditions of equilibrium for the joints provide $3n$ equations. For a simple truss, $3n = m + 6$ and the equations can be solved for m member forces and 6 support reactions.
- Equilibrium for the entire truss provides 6 additional equations which are not independent of the joint equations.

Space Trusses

Photo 6.4 – Three-dimensional or space trusses are used for broadcast and power transmission line towers, roof framing, and spacecraft applications, such as components of the *International Space Station.*

Sample Problem 6.1

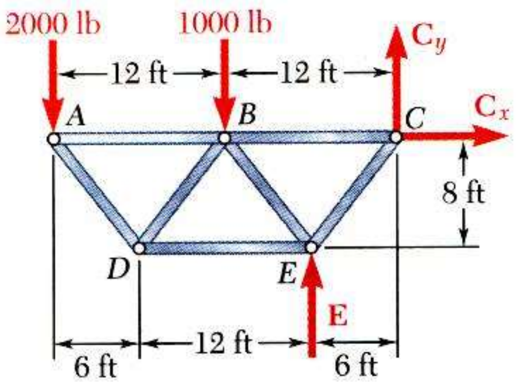


Using the method of joints, determine the force in each member of the truss.

SOLUTION:

- Based on a free-body diagram of the entire truss, solve the 3 equilibrium equations for the reactions at *E* and *C*.
- Joint *A* is subjected to only two unknown member forces. Determine these from the joint equilibrium requirements.
- In succession, determine unknown member forces at joints *D*, *B*, and *E* from joint equilibrium requirements.
- All member forces and support reactions are known at joint *C*. However, the joint equilibrium requirements may be applied to check the results.

Sample Problem 6.1



SOLUTION:

- Based on a free-body diagram of the entire truss, solve the 3 equilibrium equations for the reactions at E and C.

$$\sum M_C = 0$$
$$= (2000\,\text{lb})(24\,\text{ft}) + (1000\,\text{lb})(12\,\text{ft}) - E(6\,\text{ft})$$

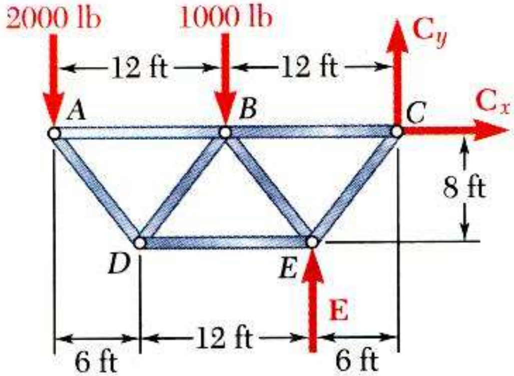
$$\boxed{E = 10{,}000\,\text{lb}\uparrow}$$

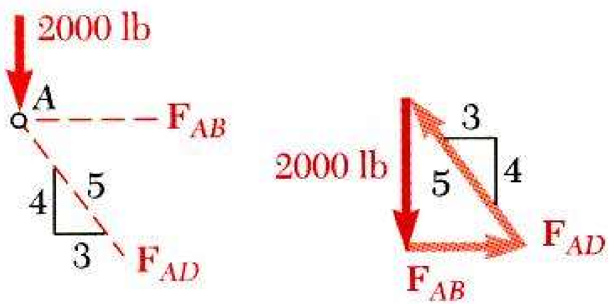
$$\sum F_x = 0 = C_x \qquad \boxed{C_x = 0}$$

$$\sum F_y = 0 = -2000\,\text{lb} - 1000\,\text{lb} + 10{,}000\,\text{lb} + C_y$$

$$\boxed{C_y = 7000\,\text{lb}\downarrow}$$

Sample Problem 6.1

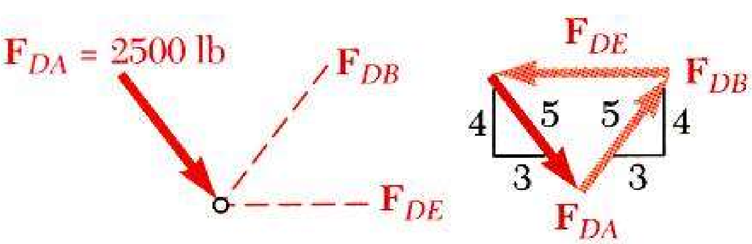




- Joint *A* is subjected to only two unknown member forces. Determine these from the joint equilibrium requirements.

$$\frac{2000\,\text{lb}}{4} = \frac{F_{AB}}{3} = \frac{F_{AD}}{5}$$

$$\boxed{\begin{aligned} F_{AB} &= 1500\,\text{lb} \;\; T \\ F_{AD} &= 2500\,\text{lb} \;\; C \end{aligned}}$$



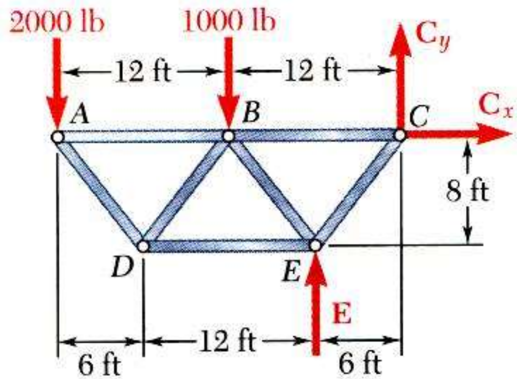
- There are now only two unknown member forces at joint D.

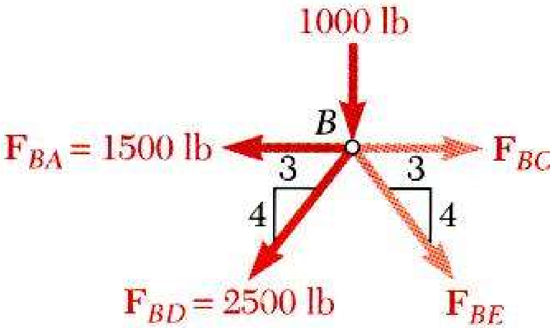
$$F_{DB} = F_{DA}$$

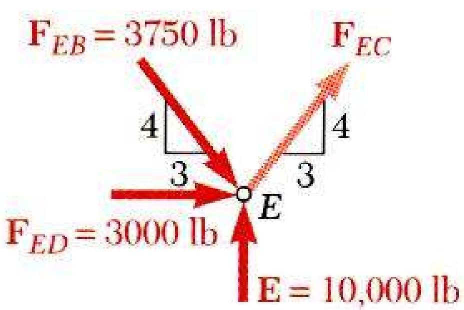
$$F_{DE} = 2\left(\tfrac{3}{5}\right)F_{DA}$$

$$\boxed{\begin{aligned} F_{DB} &= 2500\,\text{lb} \;\; T \\ F_{DE} &= 3000\,\text{lb} \;\; C \end{aligned}}$$

Sample Problem 6.1







- There are now only two unknown member forces at joint B. Assume both are in tension.

$$\sum F_y = 0 = -1000 - \tfrac{4}{5}(2500) - \tfrac{4}{5}F_{BE}$$

$$F_{BE} = -3750\,\text{lb} \qquad \boxed{F_{BE} = 3750\,\text{lb}\ C}$$

$$\sum F_x = 0 = F_{BC} - 1500 - \tfrac{3}{5}(2500) - \tfrac{3}{5}(3750)$$

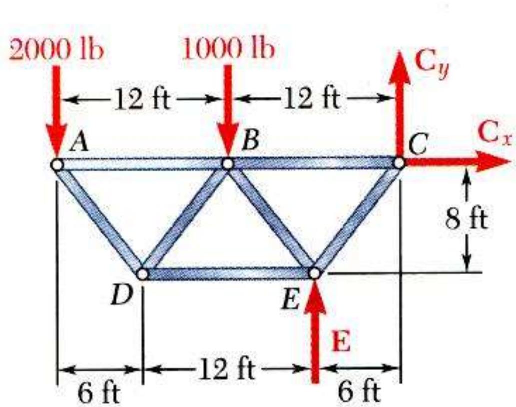
$$F_{BC} = +5250\,\text{lb} \qquad \boxed{F_{BC} = 5250\,\text{lb}\ T}$$

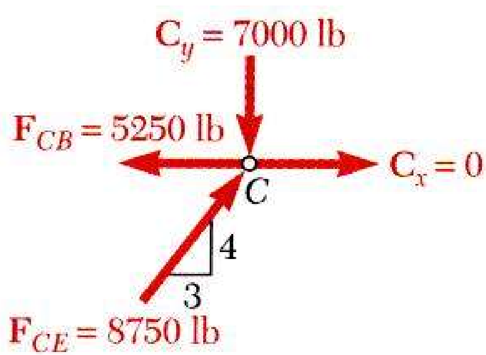
- There is one unknown member force at joint *E*. Assume the member is in tension.

$$\sum F_x = 0 = \tfrac{3}{5}F_{EC} + 3000 + \tfrac{3}{5}(3750)$$

$$F_{EC} = -8750\,\text{lb} \qquad \boxed{F_{EC} = 8750\,\text{lb}\ C}$$

Sample Problem 6.1



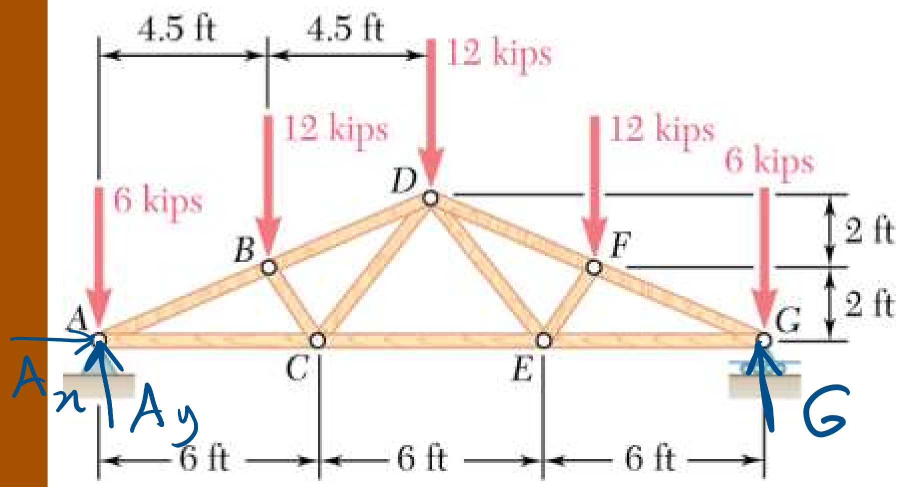


- All member forces and support reactions are known at joint C. However, the joint equilibrium requirements may be applied to check the results.

$$\sum F_x = -5250 + \tfrac{3}{5}(8750) = 0 \quad \text{(checks)}$$

$$\sum F_y = -7000 + \tfrac{4}{5}(8750) = 0 \quad \text{(checks)}$$

Problem 6.11



Using the **method of joints**, determine the force in each member of the ***Fink roof truss*** shown. State whether each member is in ***tension*** or ***compression***.

$\Sigma F_x = 0 \rightarrow A_x = 0$

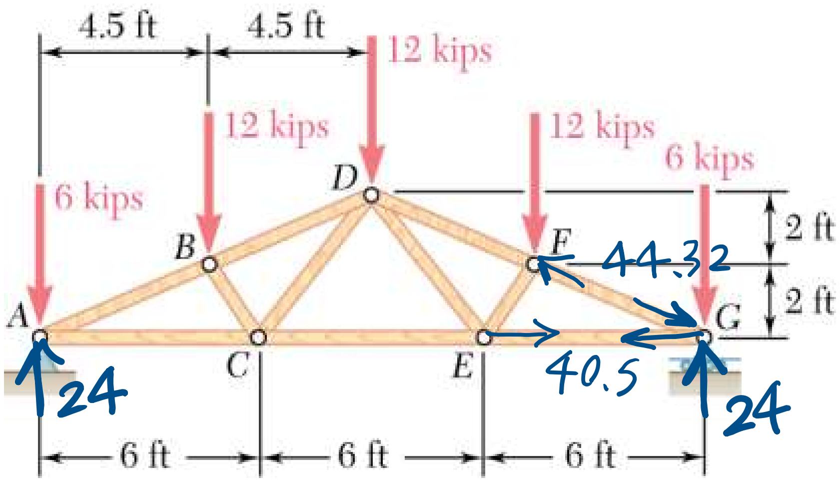
$\Sigma F_y = 0 \rightarrow -48 + G + A_y = 0$

$\Sigma M_G = 0 \rightarrow 12 \times 4.5 + 12 \times 9 + 12 \times 13.5 + 6 \times 18 - A_y \times 18 = 0$

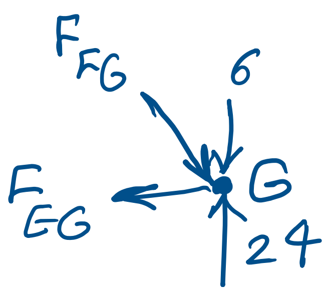
$\rightarrow A_y = 24$

$\rightarrow G = 24$

Problem 6.11



Using the **method of joints**, determine the force in each member of the ***Fink roof truss*** shown. State whether each member is in ***tension*** or ***compression***.



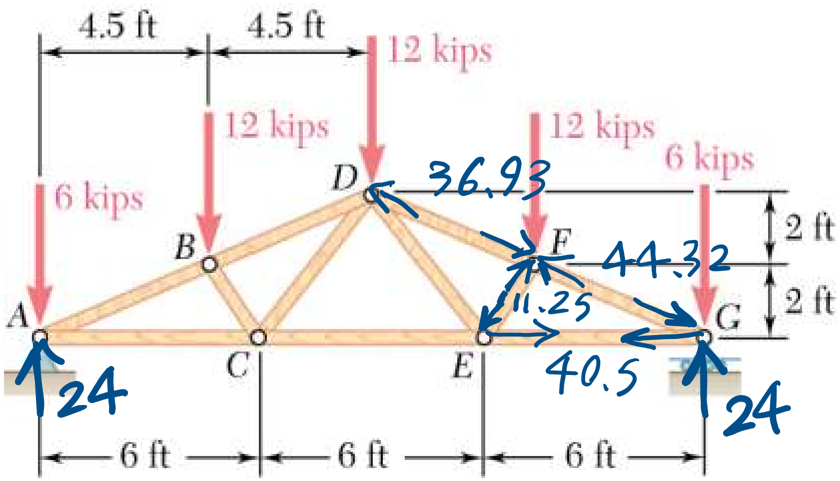
$$\Sigma F_y = 0 \rightarrow 6 + F_{FG} \times \frac{2}{\sqrt{24.25}} - 24 = 0$$

$$\rightarrow F_{FG} = 44.32$$

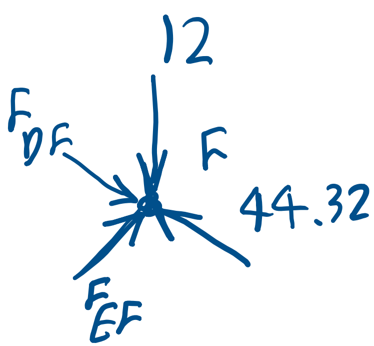
$$\Sigma F_y = 0 \rightarrow 44.32 \times \frac{4.5}{\sqrt{24.25}} - F_{EG} = 0$$

$$\rightarrow F_{EG} = 40.5$$

Problem 6.11



Using the **method of joints**, determine the force in each member of the ***Fink roof truss*** shown. State whether each member is in ***tension*** or ***compression***.

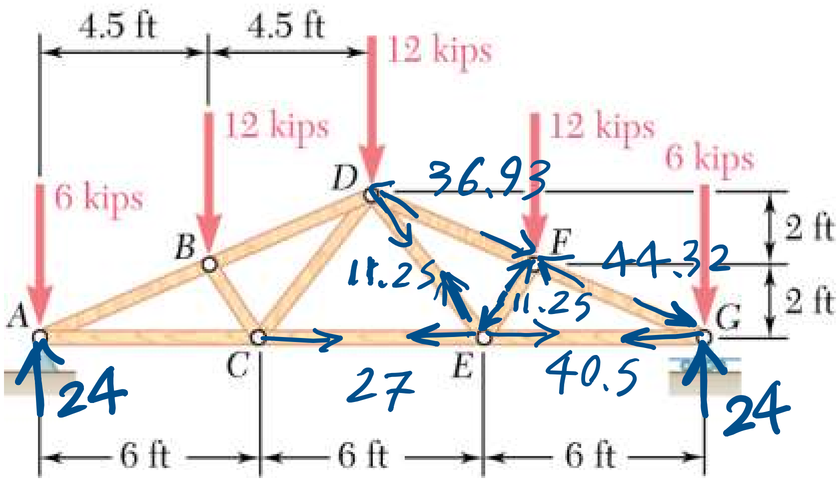


$$\Sigma F_x = 0 \rightarrow -44.32 \times \frac{4.5}{\sqrt{24.25}} + F_{DF} \times \frac{4.5}{\sqrt{24.25}} + F_{EF} \times \frac{1.5}{2.5} = 0$$

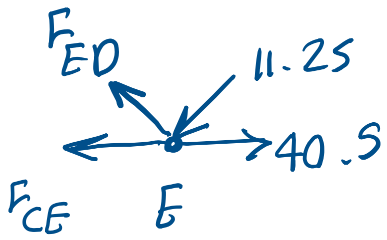
$$\Sigma F_y = 0 \rightarrow -12 + 44.32 \times \frac{2}{\sqrt{24.25}} - F_{DF} \times \frac{2}{\sqrt{24.25}} + F_{EF} \times \frac{2}{2.5} = 0$$

$$\begin{cases} F_{DF} = 36.93 \\ F_{EF} = 11.25 \end{cases}$$

Problem 6.11



Using the **method of joints**, determine the force in each member of the ***Fink roof truss*** shown. State whether each member is in ***tension*** or ***compression***.



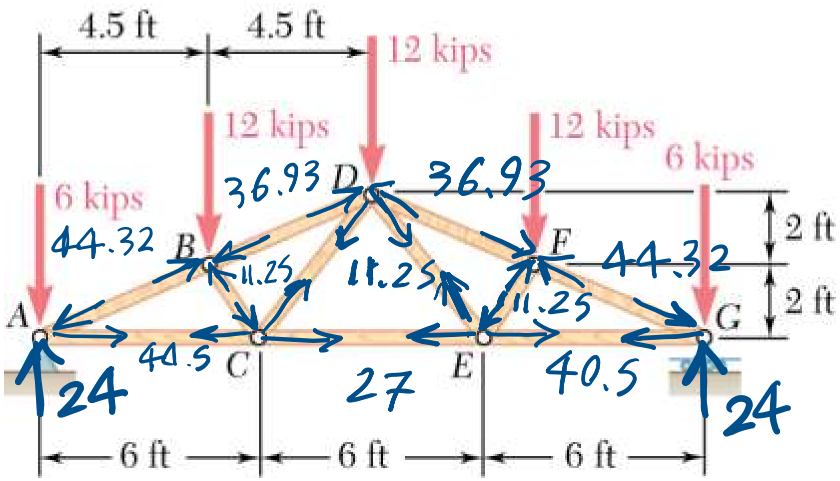
$$\Sigma F_y = 0 \rightarrow -11.25 \times \frac{2}{2.5} + F_{ED} \times \frac{4}{5} = 0$$

$$\rightarrow F_{ED} = 11.25$$

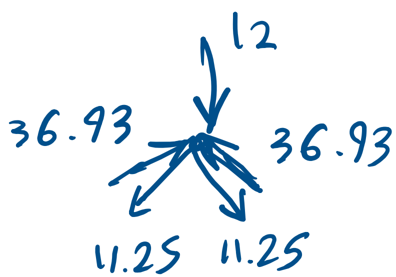
$$\Sigma F_x = 0 \rightarrow -11.25 \times \frac{1.5}{2.5} - 11.25 \times \frac{3}{5} +$$

$$40.5 - F_{CE} = 0 \rightarrow F_{CE} = 27$$

Problem 6.11



Using the **method of joints**, determine the force in each member of the ***Fink roof truss*** shown. State whether each member is in ***tension*** or ***compression***.

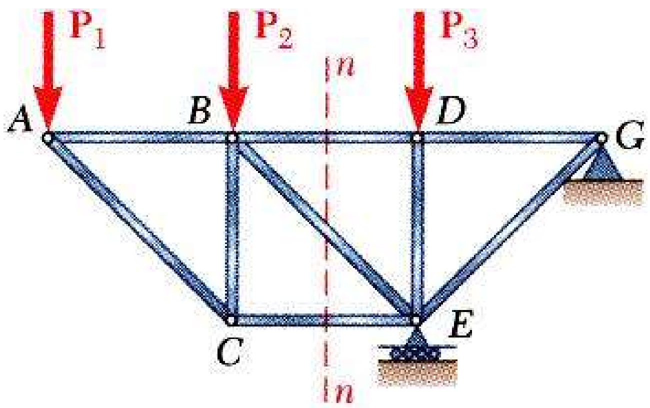


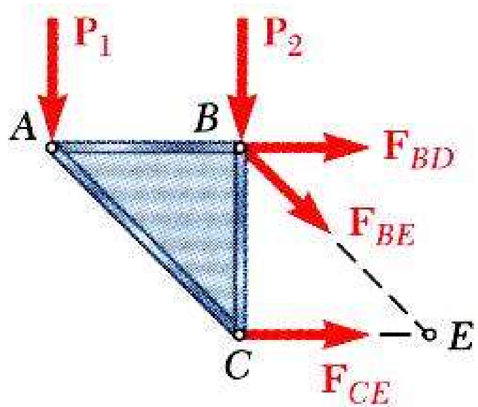
$$\Sigma F_x = 0$$

$$\Sigma F_y = 0 \rightarrow -12 + 2 \times 36.93 \times \frac{4}{\sqrt{27}} - 11.25 \times 2 \times \frac{4}{5} \simeq 0$$

دقت کنید چون محاسبات تا دو رقم اعشار بوده، باید تا دو رقم اعشار صفر شود.

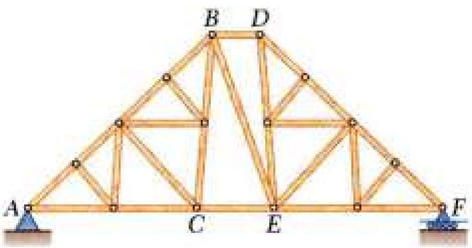
Analysis of Trusses by the Method of Sections

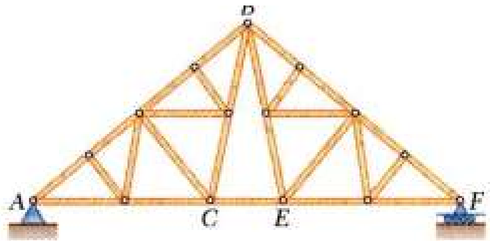




- When the force in only one member or the forces in a very few members are desired, the *method of sections* works well.
- To determine the force in member *BD, pass a section* through the truss as shown and create a free body diagram for the left side.
- With only three members cut by the section, the equations for static equilibrium may be applied to determine the unknown member forces, including F_{BD}.

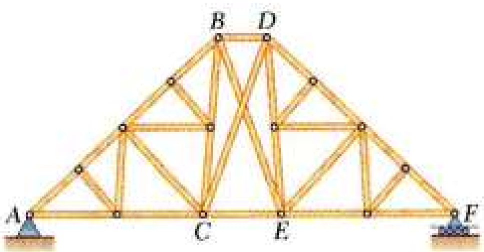
Trusses Made of Several Simple Trusses





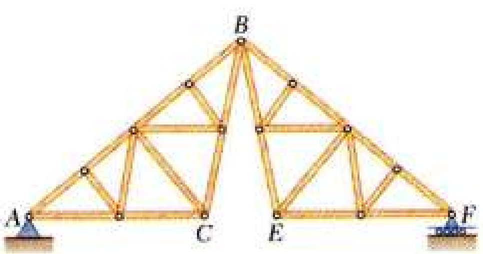
- *Compound trusses* are statically determinant, rigid, and completely constrained.

$$m = 2n - 3$$



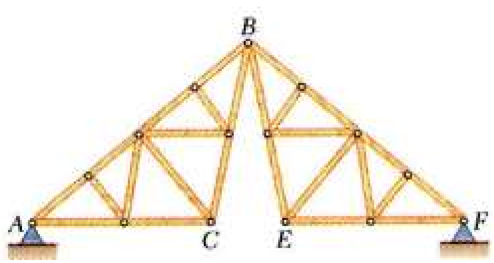
- Truss contains a *redundant member* and is *statically indeterminate.*

$$m > 2n - 3$$



non-rigid

$$m < 2n - 3$$



rigid

$$m < 2n - 4$$

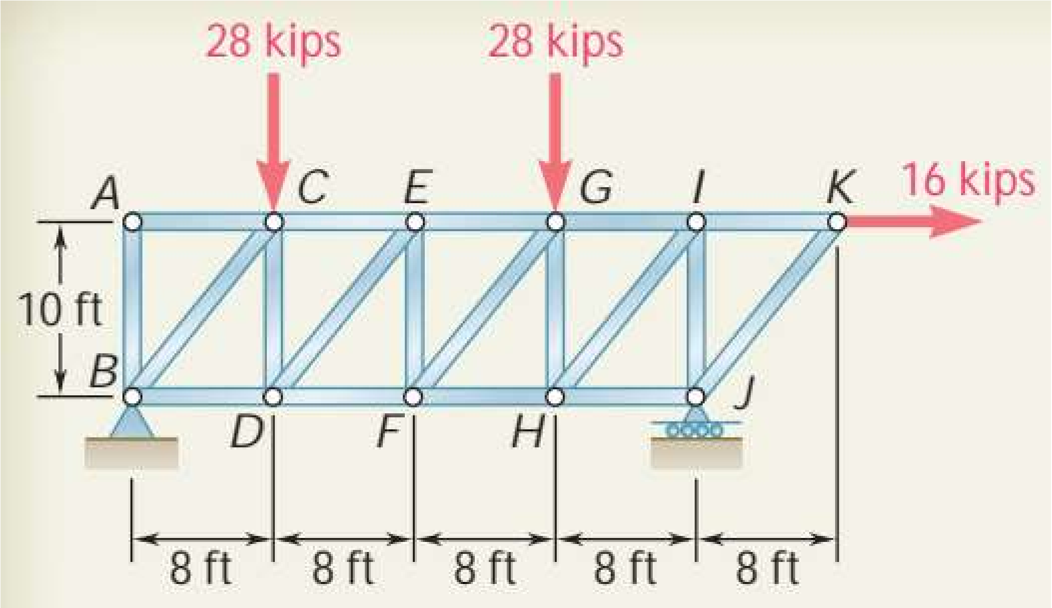
- Additional reaction forces may be necessary for a rigid truss.

- Necessary but insufficient condition for a compound truss to be statically determinant, rigid, and completely constrained,

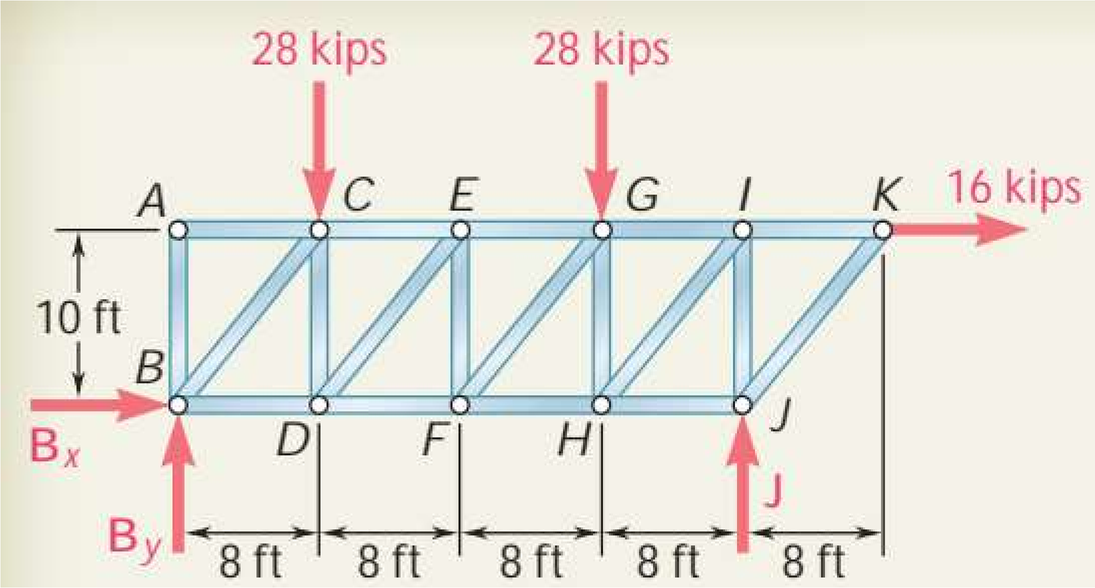
$$m + r = 2n$$

Sample Problem 6.2

Determine the force in members *EF* and *GI* of the truss shown.



Sample Problem 6.2



Sample Problem 6.2

Free-Body: Entire Truss. A free-body diagram of the entire truss is drawn; external forces acting on this free body consist of the applied loads and the reactions at B and J. We write the following equilibrium equations.

$+1\Sigma M_B = 0$:

$$-(28 \text{ kips})(8 \text{ ft}) - (28 \text{ kips})(24 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) + J(32 \text{ ft}) = 0$$

$$J = +33 \text{ kips} \qquad \mathbf{J} = 33 \text{ kips}x$$

$\overset{+}{y}\Sigma F_x = 0$: $\qquad B_x + 16 \text{ kips} = 0$

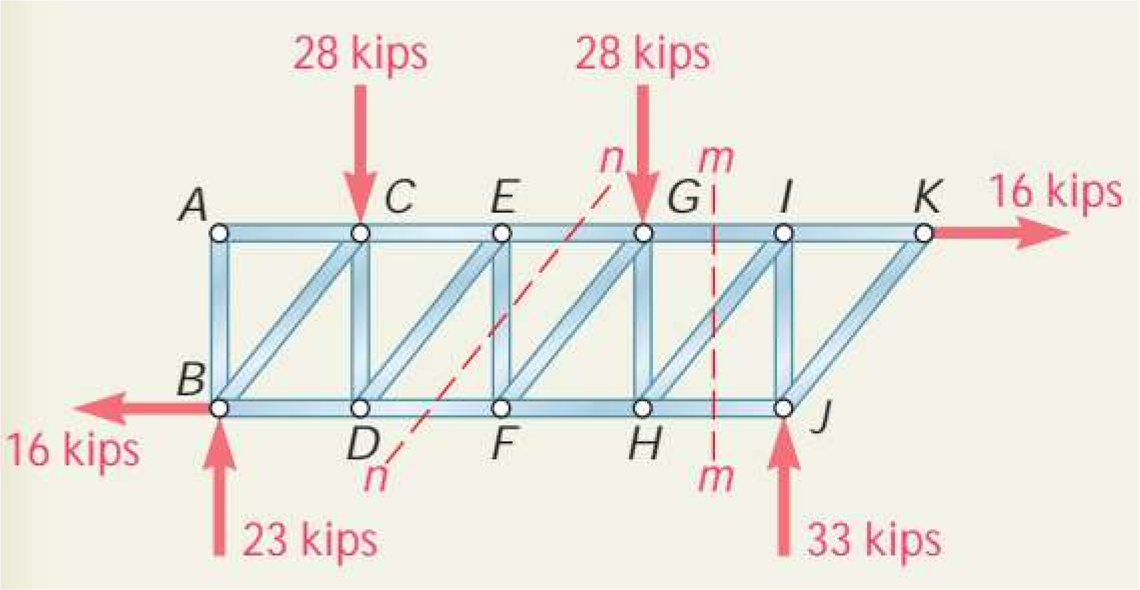
$$B_x = -16 \text{ kips} \qquad \mathbf{B}_x = 16 \text{ kips}z$$

$+1\Sigma M_J = 0$:

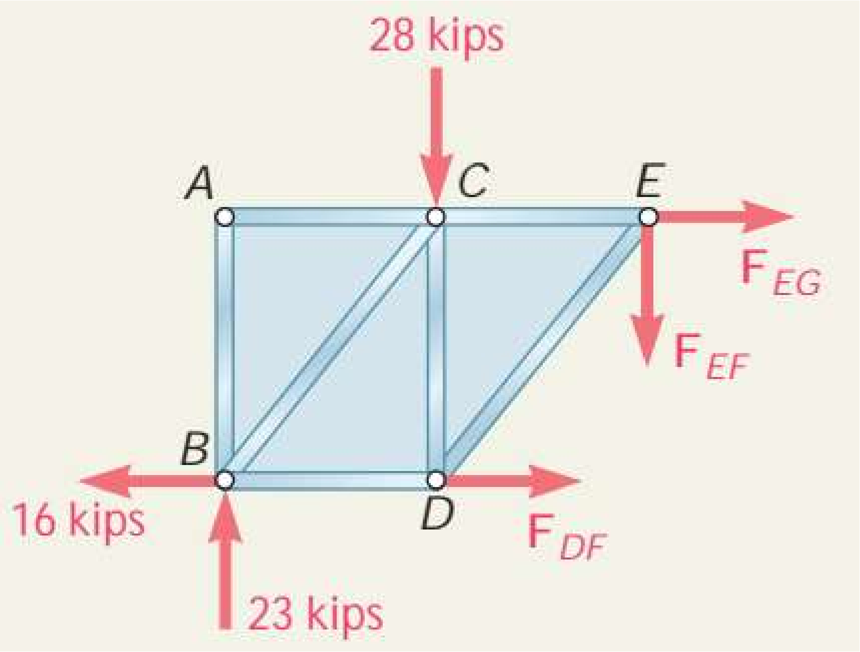
$$(28 \text{ kips})(24 \text{ ft}) + (28 \text{ kips})(8 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) - B_y(32 \text{ ft}) = 0$$

$$B_y = +23 \text{ kips} \qquad \mathbf{B}_y = 23 \text{ kips}x$$

Sample Problem 6.2



Sample Problem 6.2



Sample Problem 6.2

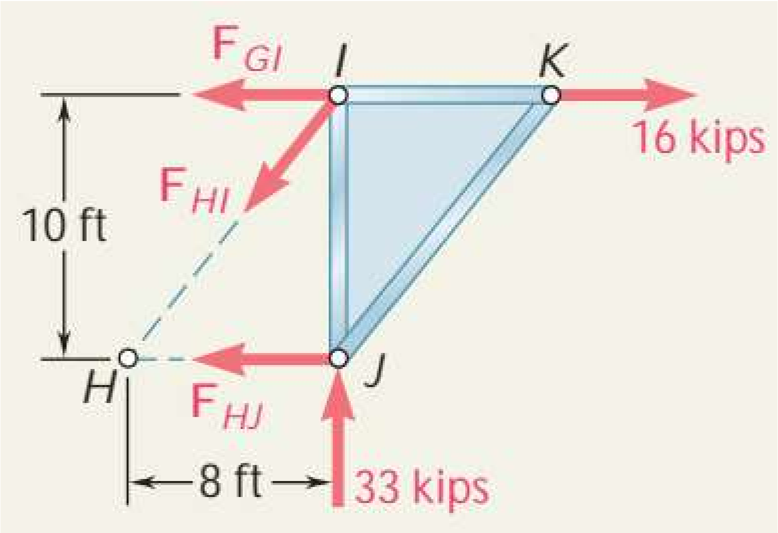
Force in Member *EF*. Section *nn* is passed through the truss so that it intersects member *EF* and only two additional members. After the intersected members have been removed, the left-hand portion of the truss is chosen as a free body. Three unknowns are involved; to eliminate the two horizontal forces, we write

$$+\!\times \Sigma F_y = 0: \qquad +23 \text{ kips} - 28 \text{ kips} - F_{EF} = 0$$
$$F_{EF} = -5 \text{ kips}$$

The sense of $\mathbf{F}_{EF}$ was chosen assuming member *EF* to be in tension; the negative sign obtained indicates that the member is in compression.

$$F_{EF} = 5 \text{ kips } C \quad \blacktriangleleft$$

Sample Problem 6.2



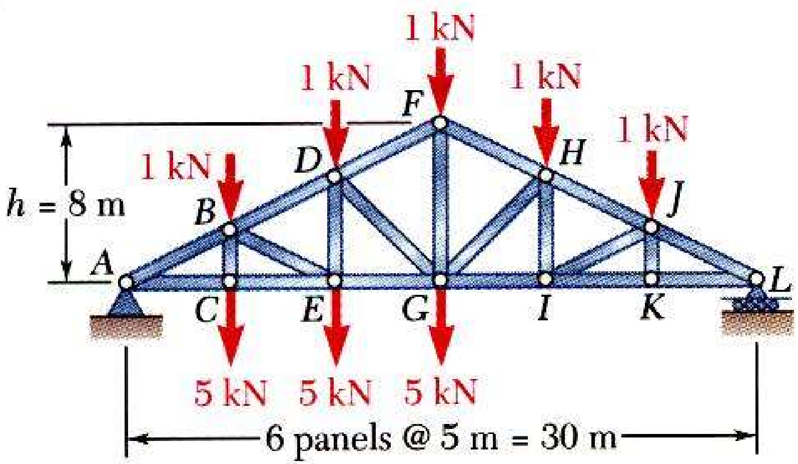
Sample Problem 6.2

Force in Member *GI*. Section *mm* is passed through the truss so that it intersects member *GI* and only two additional members. After the intersected members have been removed, we choose the right-hand portion of the truss as a free body. Three unknown forces are again involved; to eliminate the two forces passing through point *H*, we write

$$+\circlearrowleft\Sigma M_H = 0: \qquad (33 \text{ kips})(8 \text{ ft}) - (16 \text{ kips})(10 \text{ ft}) + F_{GI}(10 \text{ ft}) = 0$$

$$F_{GI} = -10.4 \text{ kips} \qquad F_{GI} = 10.4 \text{ kips } C$$

Sample Problem 6.3

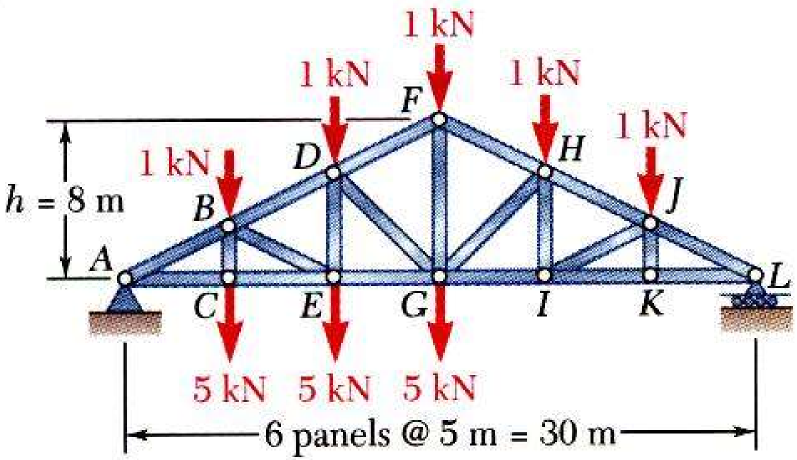


Determine the force in members *FH*, *GH*, and *GI*.

SOLUTION:

- Take the entire truss as a free body. Apply the conditions for static equilibrium to solve for the reactions at *A* and *L*.
- Pass a section through members *FH*, *GH*, and *GI* and take the right-hand section as a free body.
- Apply the conditions for static equilibrium to determine the desired member forces.

Sample Problem 6.3



SOLUTION:

- Take the entire truss as a free body. Apply the conditions for static equilibrium to solve for the reactions at *A* and *L*.

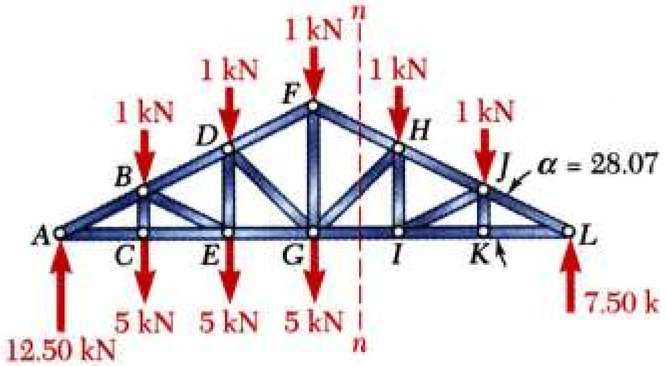
$$\sum M_A = 0 = -(5\,\text{m})(6\,\text{kN}) - (10\,\text{m})(6\,\text{kN}) - (15\,\text{m})(6\,\text{kN})$$
$$-(20\,\text{m})(1\,\text{kN}) - (25\,\text{m})(1\,\text{kN}) + (25\,\text{m})L$$

$$L = 7.5\,\text{kN} \uparrow$$

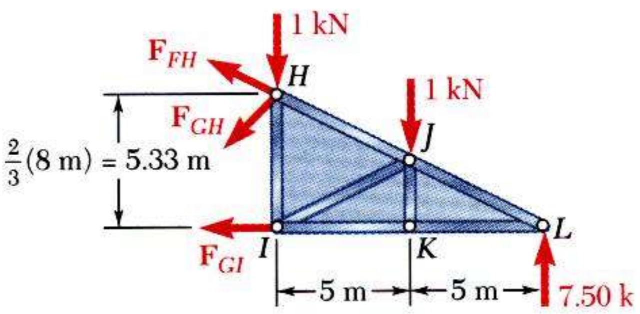
$$\sum F_y = 0 = -20\,\text{kN} + L + A$$

$$A = 12.5\,\text{kN} \uparrow$$

Sample Problem 6.3



- Pass a section through members *FH*, *GH*, and *GI* and take the right-hand section as a free body.



- Apply the conditions for static equilibrium to determine the desired member forces.

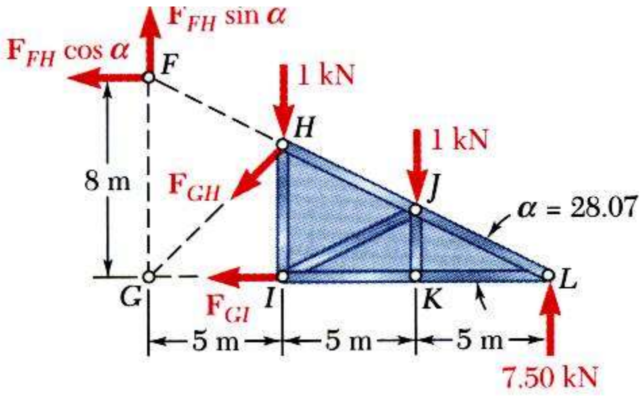
$$\sum M_H = 0$$

$$(7.50\,\text{kN})(10\,\text{m}) - (1\,\text{kN})(5\,\text{m}) - F_{GI}(5.33\,\text{m}) = 0$$

$$F_{GI} = +13.13\,\text{kN}$$

$$\boxed{F_{GI} = 13.13\,\text{kN} \;\; T}$$

Sample Problem 6.3



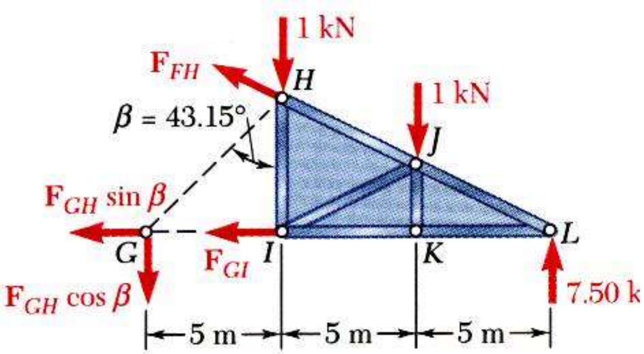
$$\tan\alpha = \frac{FG}{GL} = \frac{8\text{ m}}{15\text{ m}} = 0.5333 \qquad \alpha = 28.07°$$

$$\sum M_G = 0$$

$$(7.5\text{ kN})(15\text{ m}) - (1\text{ kN})(10\text{ m}) - (1\text{ kN})(5\text{ m}) + (F_{FH}\cos\alpha)(8\text{ m}) = 0$$

$$F_{FH} = -13.82\text{ kN}$$

$$\boxed{F_{FH} = 13.82\text{ kN} \;\; C}$$



$$\tan\beta = \frac{GI}{HI} = \frac{5\text{ m}}{\frac{2}{3}(8\text{ m})} = 0.9375 \qquad \beta = 43.15°$$

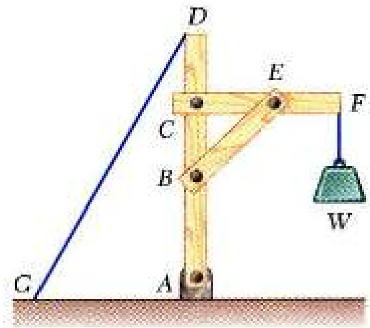
$$\sum M_L = 0$$

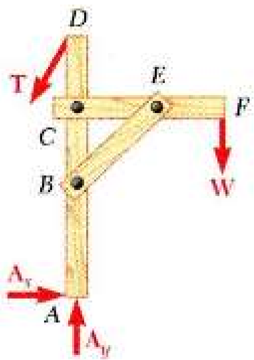
$$(1\text{ kN})(10\text{ m}) + (1\text{ kN})(5\text{ m}) + (F_{GH}\cos\beta)(10\text{ m}) = 0$$

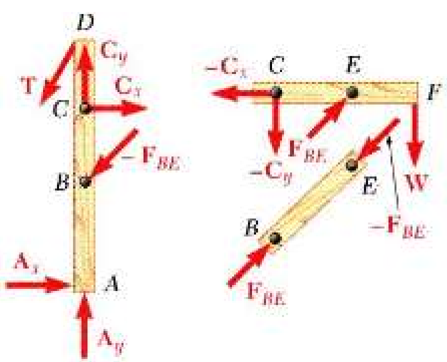
$$F_{GH} = -1.371\text{ kN}$$

$$\boxed{F_{GH} = 1.371\text{ kN} \;\; C}$$

Analysis of Frames

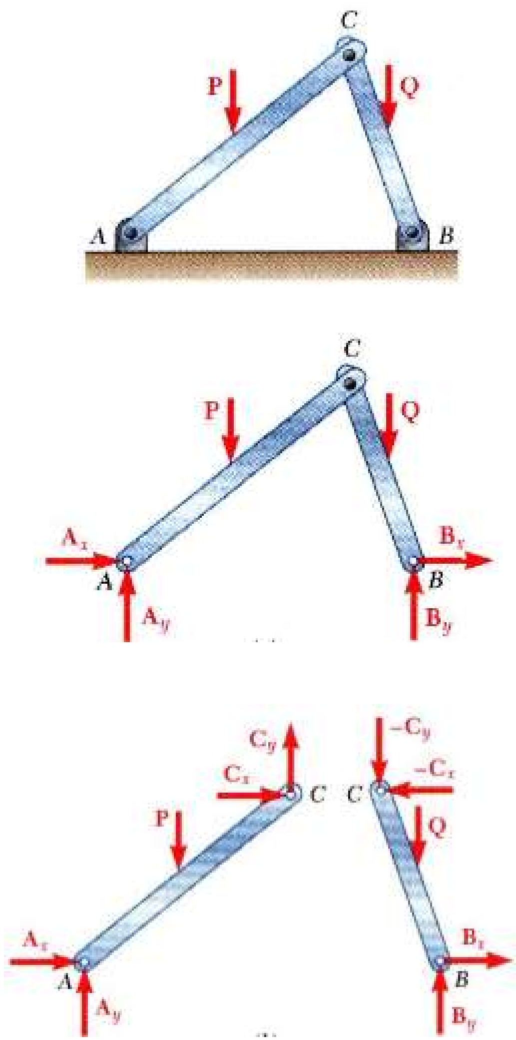






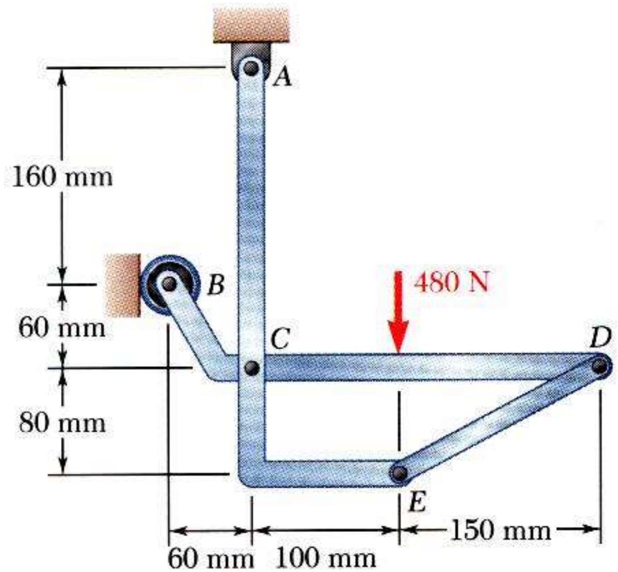
- *Frames* and *machines* are structures with at least one *multiforce* member. Frames are designed to support loads and are usually stationary. Machines contain moving parts and are designed to transmit and modify forces.
- A free body diagram of the complete frame is used to determine the external forces acting on the frame.
- Internal forces are determined by dismembering the frame and creating free-body diagrams for each component.
- Forces on two force members have known lines of action but unknown magnitude and sense.
- Forces on multiforce members have unknown magnitude and line of action. They must be represented with two unknown components.
- Forces between connected components are equal, have the same line of action, and opposite sense.

Frames Which Cease To Be Rigid When Detached From Their Supports



- Some frames may collapse if removed from their supports. Such frames can not be treated as rigid bodies.
- A free-body diagram of the complete frame indicates four unknown force components which can not be determined from the three equilibrium conditions.
- The frame must be considered as two distinct, but related, rigid bodies.
- With equal and opposite reactions at the contact point between members, the two free-body diagrams indicate 6 unknown force components.
- Equilibrium requirements for the two rigid bodies yield 6 independent equations.

Sample Problem 6.4

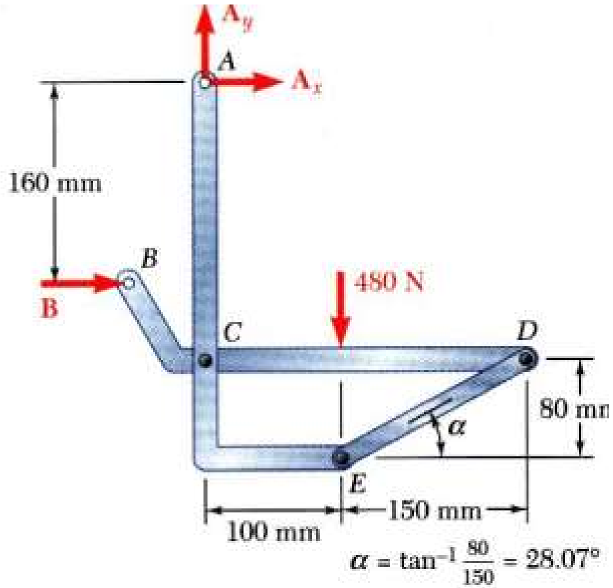


Members *ACE* and *BCD* are connected by a pin at *C* and by the link *DE*. For the loading shown, determine the force in link *DE* and the components of the force exerted at *C* on member *BCD*.

SOLUTION:

- Create a free-body diagram for the complete frame and solve for the support reactions.
- Define a free-body diagram for member *BCD*. The force exerted by the link *DE* has a known line of action but unknown magnitude. It is determined by summing moments about *C*.
- With the force on the link *DE* known, the sum of forces in the *x* and *y* directions may be used to find the force components at *C*.
- With member *ACE* as a free-body, check the solution by summing moments about *A*.

Sample Problem 6.4



SOLUTION:

- Create a free-body diagram for the complete frame and solve for the support reactions.

$$\sum F_y = 0 = A_y - 480\text{ N}$$

$$\boxed{A_y = 480\text{ N} \uparrow}$$

$$\sum M_A = 0 = -(480\text{ N})(100\text{ mm}) + B(160\text{ mm})$$

$$\boxed{B = 300\text{ N} \rightarrow}$$

$$\sum F_x = 0 = B + A_x$$

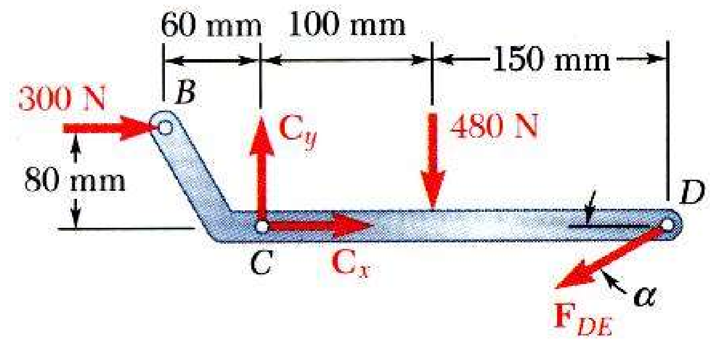
$$\boxed{A_x = -300\text{ N} \leftarrow}$$

Note:

$$\alpha = \tan^{-1}\frac{80}{150} = 28.07°$$

Sample Problem 6.4

- Define a free-body diagram for member *BCD*. The force exerted by the link *DE* has a known line of action but unknown magnitude. It is determined by summing moments about *C*.



$$\sum M_C = 0 = (F_{DE}\sin\alpha)(250\text{ mm}) + (300\text{ N})(80\text{ mm}) + (480\text{ N})(100\text{ mm})$$

$$F_{DE} = -561\text{ N} \qquad \boxed{F_{DE} = 561\text{ N } C}$$

- Sum of forces in the *x* and *y* directions may be used to find the force components at *C*.

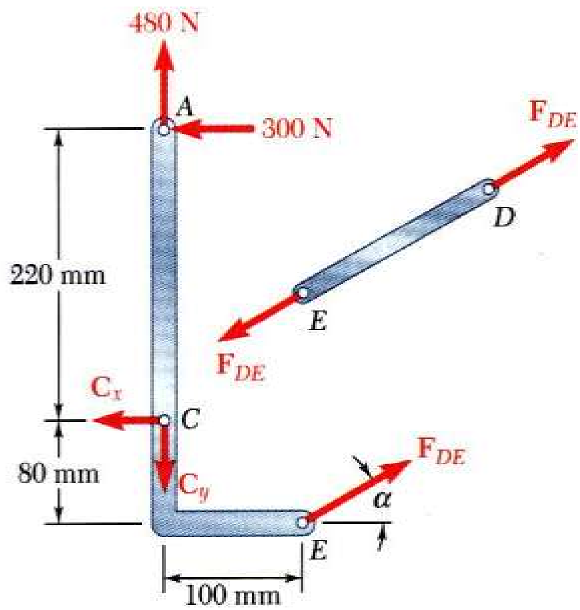
$$\sum F_x = 0 = C_x - F_{DE}\cos\alpha + 300\text{ N}$$

$$0 = C_x - (-561\text{ N})\cos\alpha + 300\text{ N} \qquad \boxed{C_x = -795\text{ N}}$$

$$\sum F_y = 0 = C_y - F_{DE}\sin\alpha - 480\text{ N}$$

$$0 = C_y - (-561\text{ N})\sin\alpha - 480\text{ N} \qquad \boxed{C_y = 216\text{ N}}$$

Sample Problem 6.4



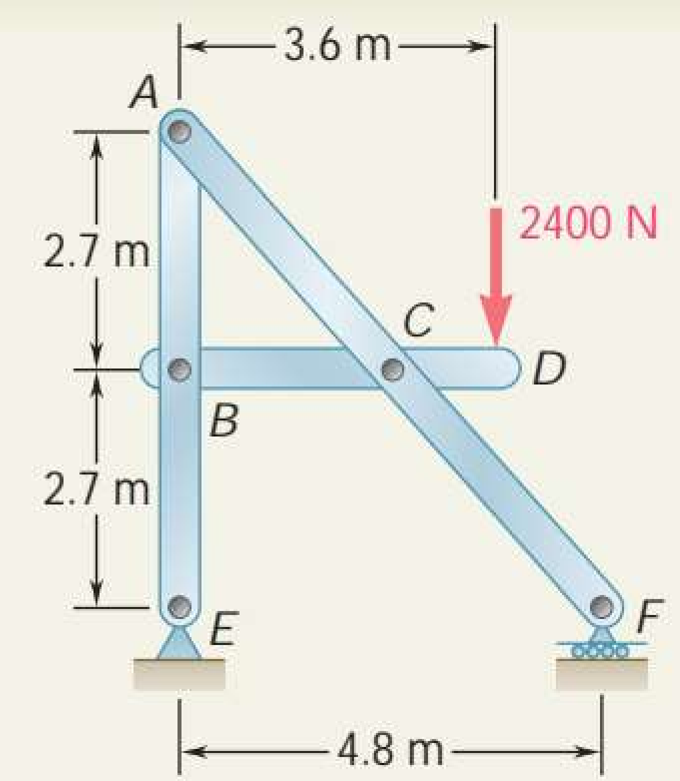
- With member *ACE* as a free-body, check the solution by summing moments about *A*.

$$\sum M_A = (F_{DE}\cos\alpha)(300\,\text{mm}) + (F_{DE}\sin\alpha)(100\,\text{mm}) - C_x(220\,\text{mm})$$
$$= (-561\cos\alpha)(300\,\text{mm}) + (-561\sin\alpha)(100\,\text{mm}) - (-795)(220\,\text{mm}) = 0$$

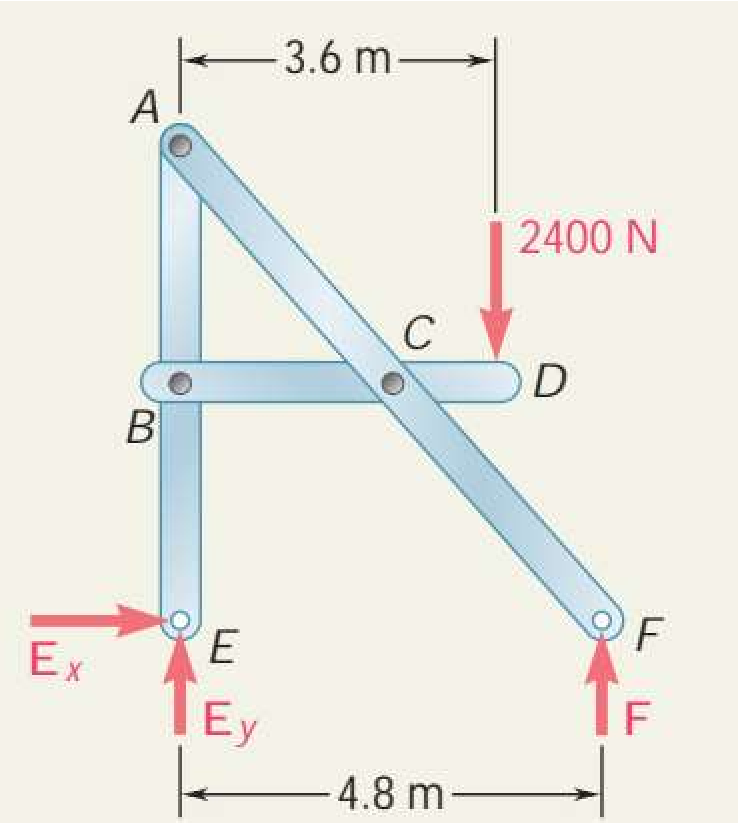
(checks)

Sample Problem 6.5

Determine the components of the forces acting on each member of the frame shown.



Sample Problem 6.5



Sample Problem 6.5

Free Body: Entire Frame. Since the external reactions involve only three unknowns, we compute the reactions by considering the free-body diagram of the entire frame.

$+1\Sigma M_E = 0: \quad -(2400\text{ N})(3.6\text{ m}) + F(4.8\text{ m}) = 0$

$F = +1800\text{ N} \qquad \mathbf{F} = 1800\text{ Nx}$

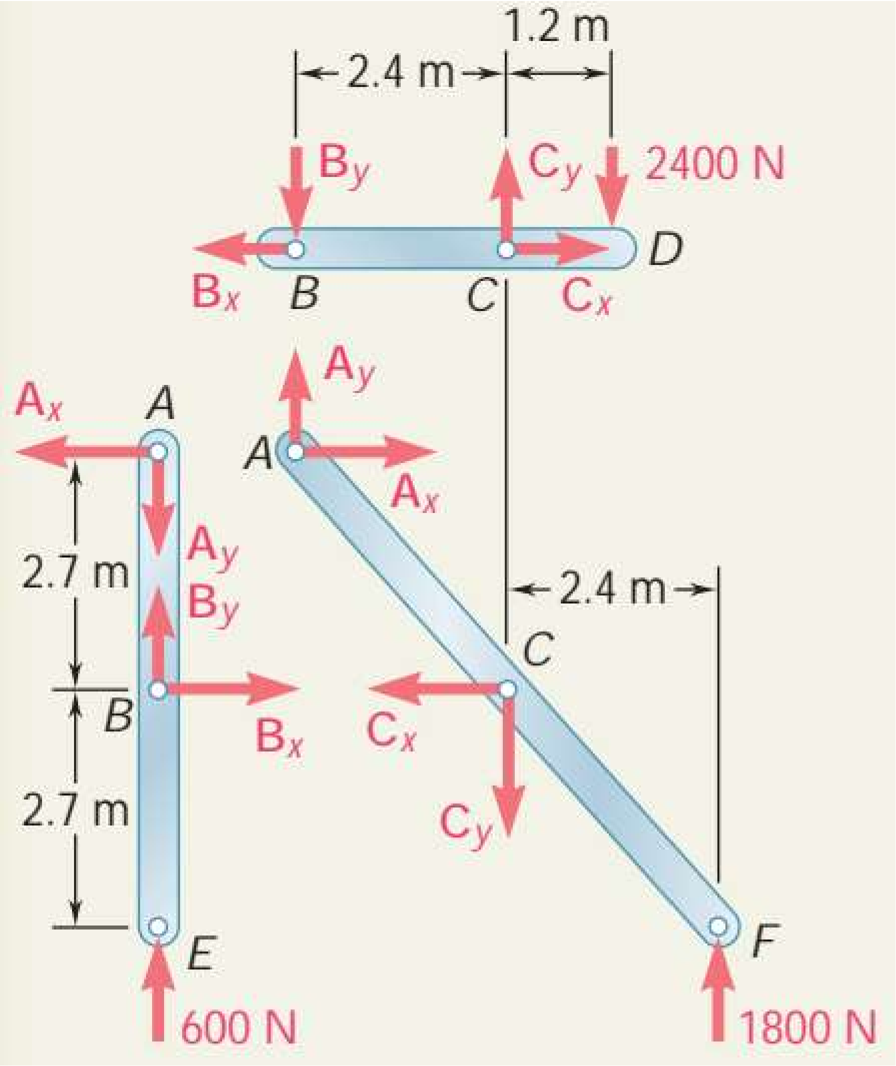
$+\mathrm{x}\,\Sigma F_y = 0: \quad -2400\text{ N} + 1800\text{ N} + E_y = 0$

$E_y = +600\text{ N} \qquad \mathbf{E}_y = 600\text{ Nx}$

$\overset{+}{\mathrm{y}}\,\Sigma F_x = 0: \qquad \mathbf{E}_x = 0$

Members. The frame is now dismembered; since only two members are connected at each joint, equal and opposite components are shown on each member at each joint.

Sample Problem 6.5



Sample Problem 6.5

Free Body: Member BCD

$$+\circlearrowleft \Sigma M_B = 0: \quad -(2400 \text{ N})(3.6 \text{ m}) + C_y(2.4 \text{ m}) = 0 \qquad C_y = +3600 \text{ N}$$
$$+\circlearrowleft \Sigma M_C = 0: \quad -(2400 \text{ N})(1.2 \text{ m}) + B_y(2.4 \text{ m}) = 0 \qquad B_y = +1200 \text{ N}$$
$$\xrightarrow{+} \Sigma F_x = 0: \quad -B_x + C_x = 0$$

We note that neither B_x nor C_x can be obtained by considering only member BCD. The positive values obtained for B_y and C_y indicate that the force components $\mathbf{B}_y$ and $\mathbf{C}_y$ are directed as assumed.

Free Body: Member ABE

$$+\circlearrowleft \Sigma M_A = 0: \quad B_x(2.7 \text{ m}) = 0 \qquad B_x = 0$$
$$\xrightarrow{+} \Sigma F_x = 0: \quad +B_x - A_x = 0 \qquad A_x = 0$$
$$+\uparrow \Sigma F_y = 0: \quad -A_y + B_y + 600 \text{ N} = 0$$
$$-A_y + 1200 \text{ N} + 600 \text{ N} = 0 \qquad A_y = +1800 \text{ N}$$

Sample Problem 6.5

Free Body: Member BCD. Returning now to member *BCD*, we write

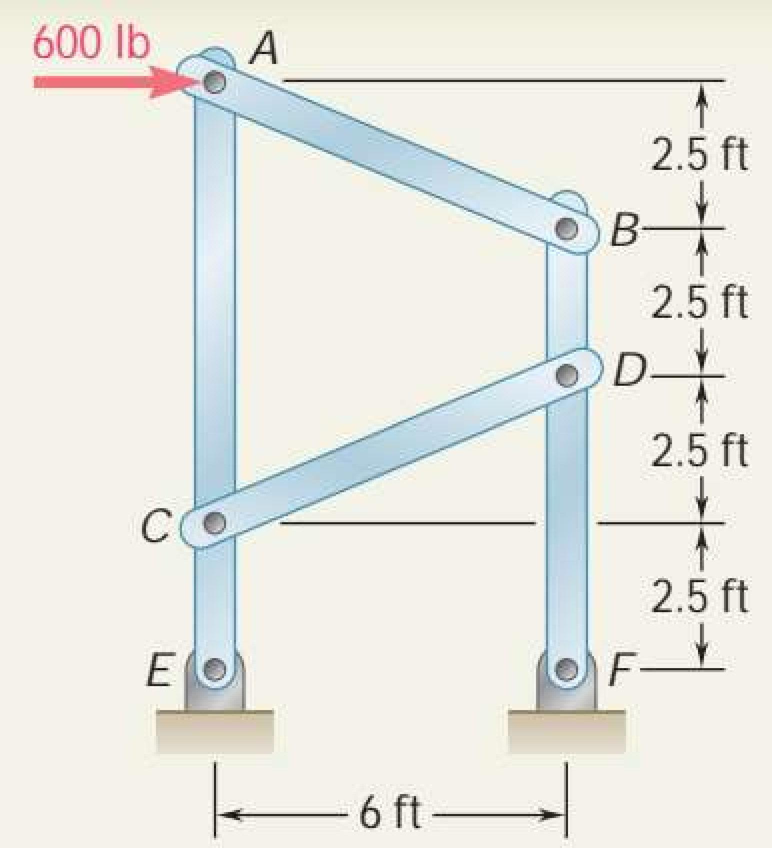
$$\xrightarrow{+}\Sigma F_x = 0: \qquad -B_x + C_x = 0 \qquad 0 + C_x = 0 \qquad\qquad C_x = 0 \blacktriangleleft$$

Free Body: Member ACF (Check). All unknown components have now been found; to check the results, we verify that member *ACF* is in equilibrium.

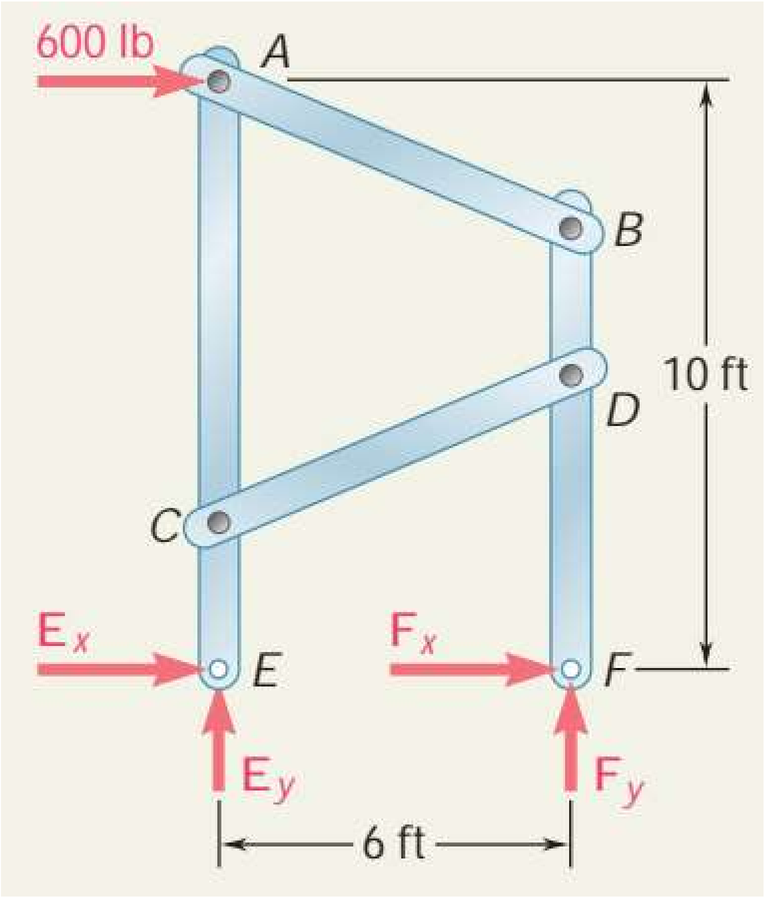
$$+\circlearrowleft\Sigma M_C = (1800 \text{ N})(2.4 \text{ m}) - A_y(2.4 \text{ m}) - A_x(2.7 \text{ m})$$
$$= (1800 \text{ N})(2.4 \text{ m}) - (1800 \text{ N})(2.4 \text{ m}) - 0 = 0 \qquad \text{(checks)}$$

Sample Problem 6.6

A 600-lb horizontal force is applied to pin *A* of the frame shown. Determine the forces acting on the two vertical members of the frame.



Sample Problem 6.6



Sample Problem 6.6

Free Body: Entire Frame. The entire frame is chosen as a free body; although the reactions involve four unknowns, $\mathbf{E}_y$ and $\mathbf{F}_y$ may be determined by writing

$+1\Sigma M_E = 0$: $\quad -(600 \text{ lb})(10 \text{ ft}) + F_y(6 \text{ ft}) = 0$

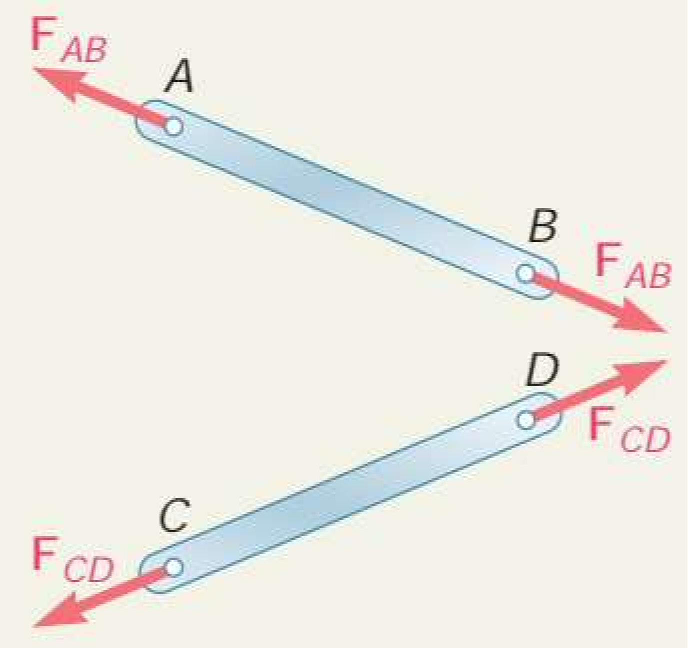
$F_y = +1000 \text{ lb}$ $\qquad \mathbf{F}_y = 1000 \text{ lbx}$ ◀

$+\text{x}\, \Sigma F_y = 0$: $\quad E_y + F_y = 0$

$E_y = -1000 \text{ lb}$ $\qquad \mathbf{E}_y = 1000 \text{ lbw}$ ◀

Members. The equations of equilibrium of the entire frame are not sufficient to determine $\mathbf{E}_x$ and $\mathbf{F}_x$. The free-body diagrams of the various members must now be considered in order to proceed with the solution. In dismembering the frame we will assume that pin *A* is attached to the multiforce member *ACE* and, thus, that the 600-lb force is applied to that member. We also note that *AB* and *CD* are two-force members.

Sample Problem 6.6



Free Body: Member ACE

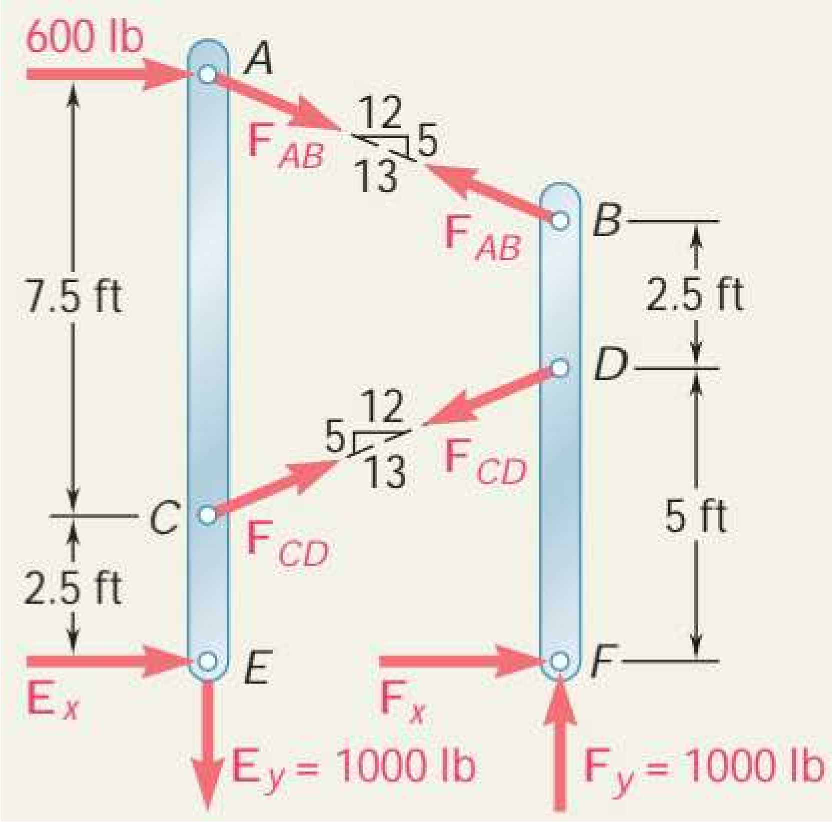
$$+\uparrow \Sigma F_y = 0: \qquad -\tfrac{5}{13}F_{AB} + \tfrac{5}{13}F_{CD} - 1000 \text{ lb} = 0$$

$$+\circlearrowleft \Sigma M_E = 0: \qquad -(600 \text{ lb})(10 \text{ ft}) - (\tfrac{12}{13}F_{AB})(10 \text{ ft}) - (\tfrac{12}{13}F_{CD})(2.5 \text{ ft}) = 0$$

Solving these equations simultaneously, we find

$$F_{AB} = -1040 \text{ lb} \qquad F_{CD} = +1560 \text{ lb}$$

Sample Problem 6.6



Sample Problem 6.6

The signs obtained indicate that the sense assumed for F_{CD} was correct and the sense for F_{AB} incorrect. Summing now x components,

$$\xrightarrow{+} \Sigma F_x = 0: \qquad 600 \text{ lb} + \tfrac{12}{13}(-1040 \text{ lb}) + \tfrac{12}{13}(+1560 \text{ lb}) + E_x = 0$$

$$E_x = -1080 \text{ lb} \qquad \mathbf{E}_x = 1080 \text{ lb} \leftarrow \blacktriangleleft$$

Free Body: Entire Frame. Since $\mathbf{E}_x$ has been determined, we can return to the free-body diagram of the entire frame and write

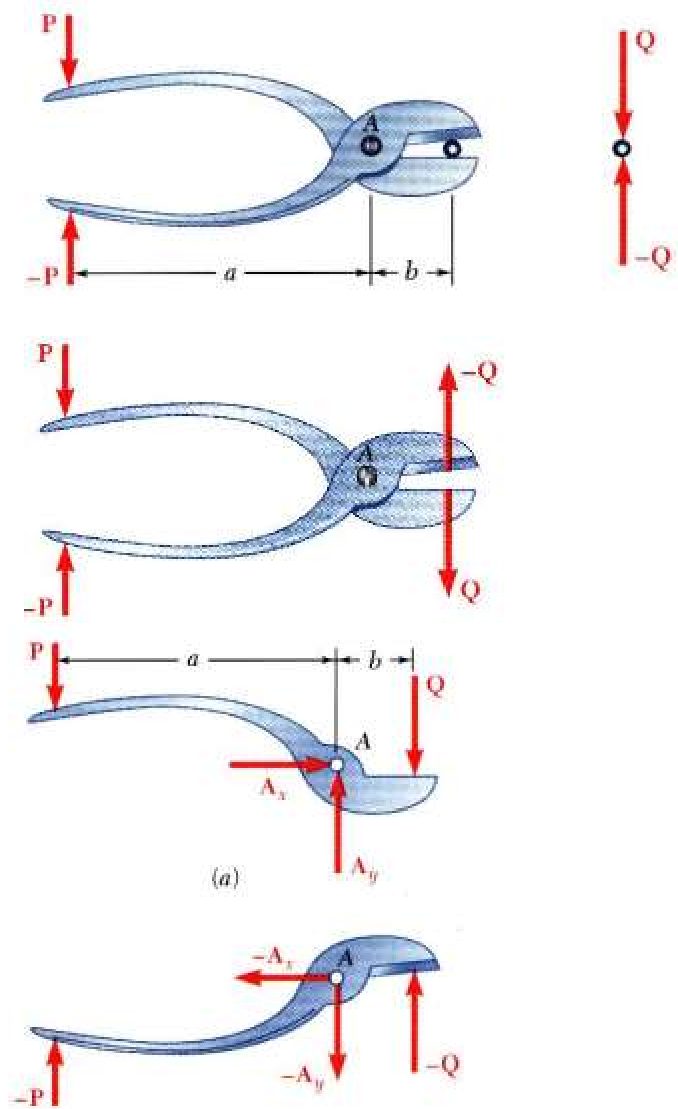
$$\xrightarrow{+} \Sigma F_x = 0: \qquad 600 \text{ lb} - 1080 \text{ lb} + F_x = 0$$

$$F_x = +480 \text{ lb} \qquad \mathbf{F}_x = 480 \text{ lb} \rightarrow \blacktriangleleft$$

Free Body: Member BDF (Check). We can check our computations by verifying that the equation $\Sigma M_B = 0$ is satisfied by the forces acting on member BDF.

$$
\begin{aligned}
+\circlearrowleft \Sigma M_B &= -\left(\tfrac{12}{13}F_{CD}\right)(2.5\text{ ft}) + (F_x)(7.5\text{ ft}) \\
&= -\tfrac{12}{13}(1560\text{ lb})(2.5\text{ ft}) + (480\text{ lb})(7.5\text{ ft}) \\
&= -3600\text{ lb}\cdot\text{ft} + 3600\text{ lb}\cdot\text{ft} = 0 \qquad \text{(checks)}
\end{aligned}
$$

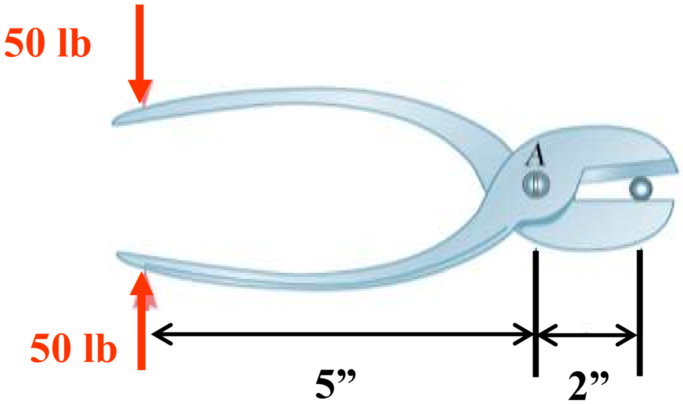
Machines



- Machines are structures designed to transmit and modify forces. Their main purpose is to transform *input forces* into *output forces*.
- Given the magnitude of $\boldsymbol{P}$, determine the magnitude of $\boldsymbol{Q}$.
- Create a free-body diagram of the complete machine, including the reaction that the wire exerts.
- The machine is a nonrigid structure. Use one of the components as a free-body.
- Taking moments about A,

$$\sum M_A = 0 = aP - bQ \qquad Q = \frac{a}{b}P$$

Example: Simple pliers



For the **machine** (simple pliers) shown, determine the magnitude of the gripping forces when two 50-lb forces are applied as shown.

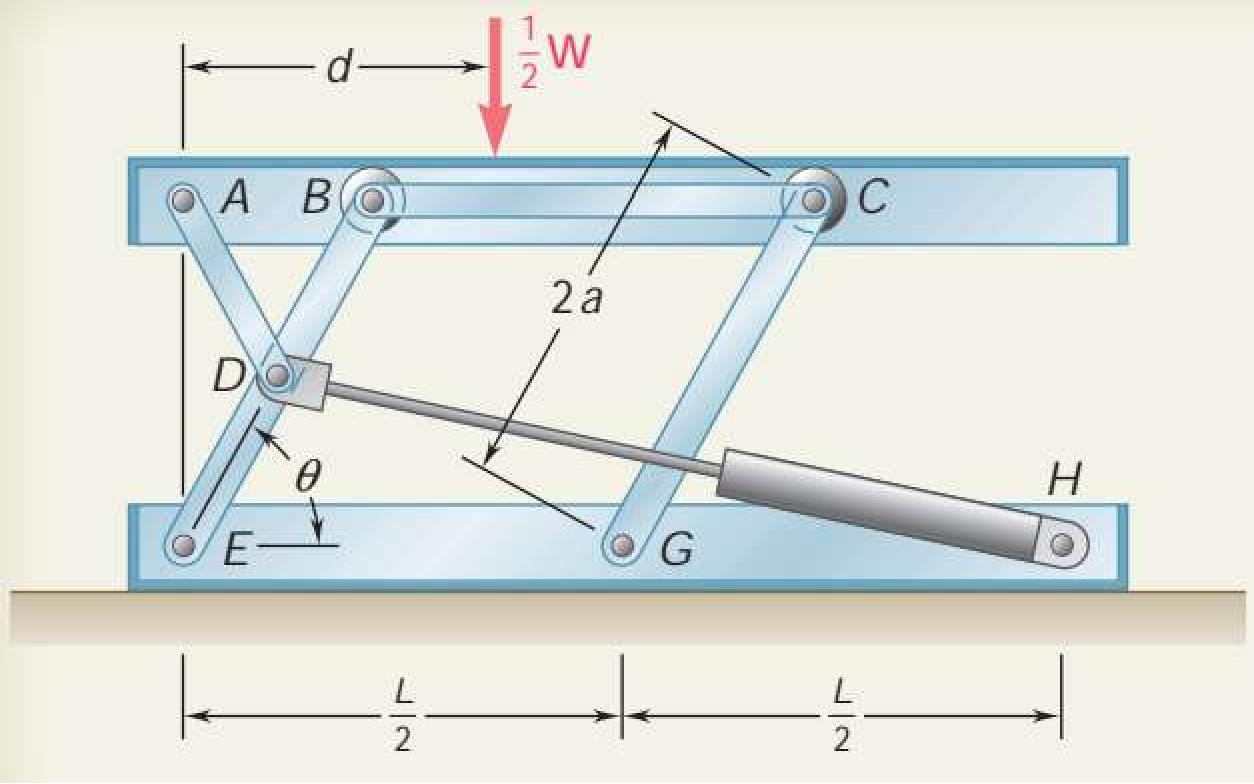
Also determine the ***mechanical advantage***.

$$\boxed{\text{Mechanical advantage} = \frac{\text{Output force}}{\text{Input force}}}$$

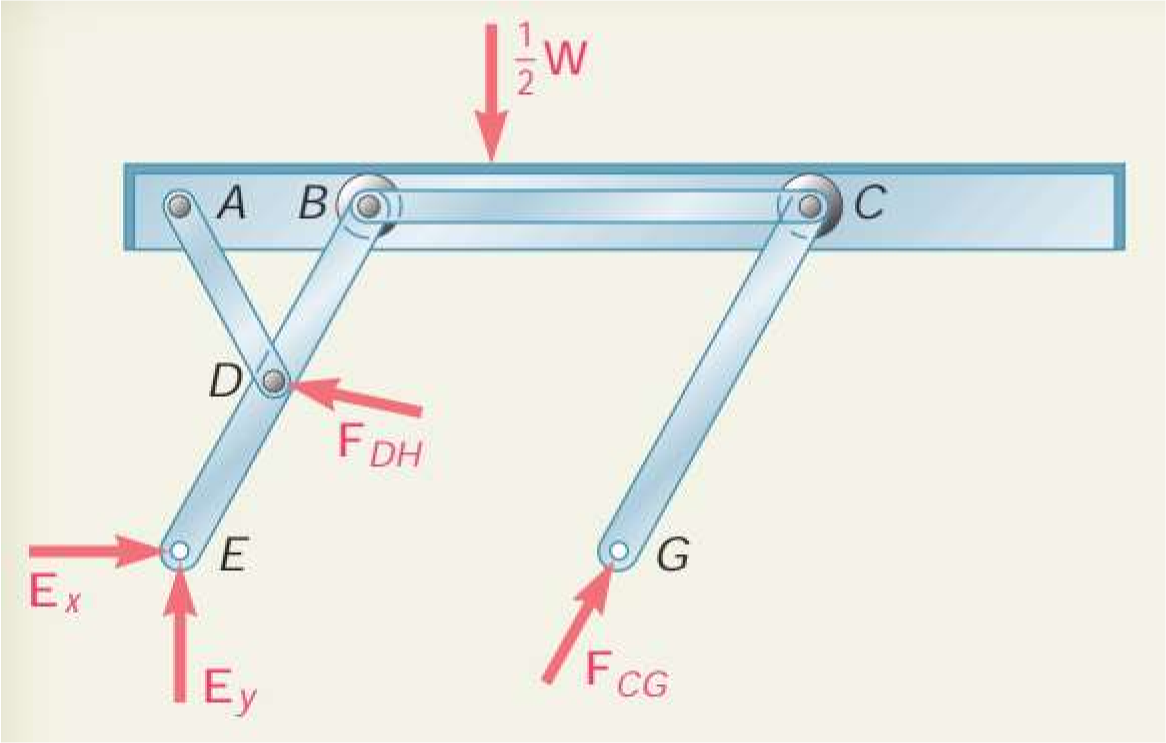
Sample Problem 6.7

A hydraulic-lift table is used to raise a 1000-kg crate. It consists of a platform and two identical linkages on which hydraulic cylinders exert equal forces. (Only one linkage and one cylinder are shown.) Members *EDB* and *CG* are each of length $2a$, and member *AD* is pinned to the midpoint of *EDB*. If the crate is placed on the table, so that half of its weight is supported by the system shown, determine the force exerted by each cylinder in raising the crate for $\theta = 60°$, $a = 0.70$ m, and $L = 3.20$ m. Show that the result obtained is independent of the distance d.

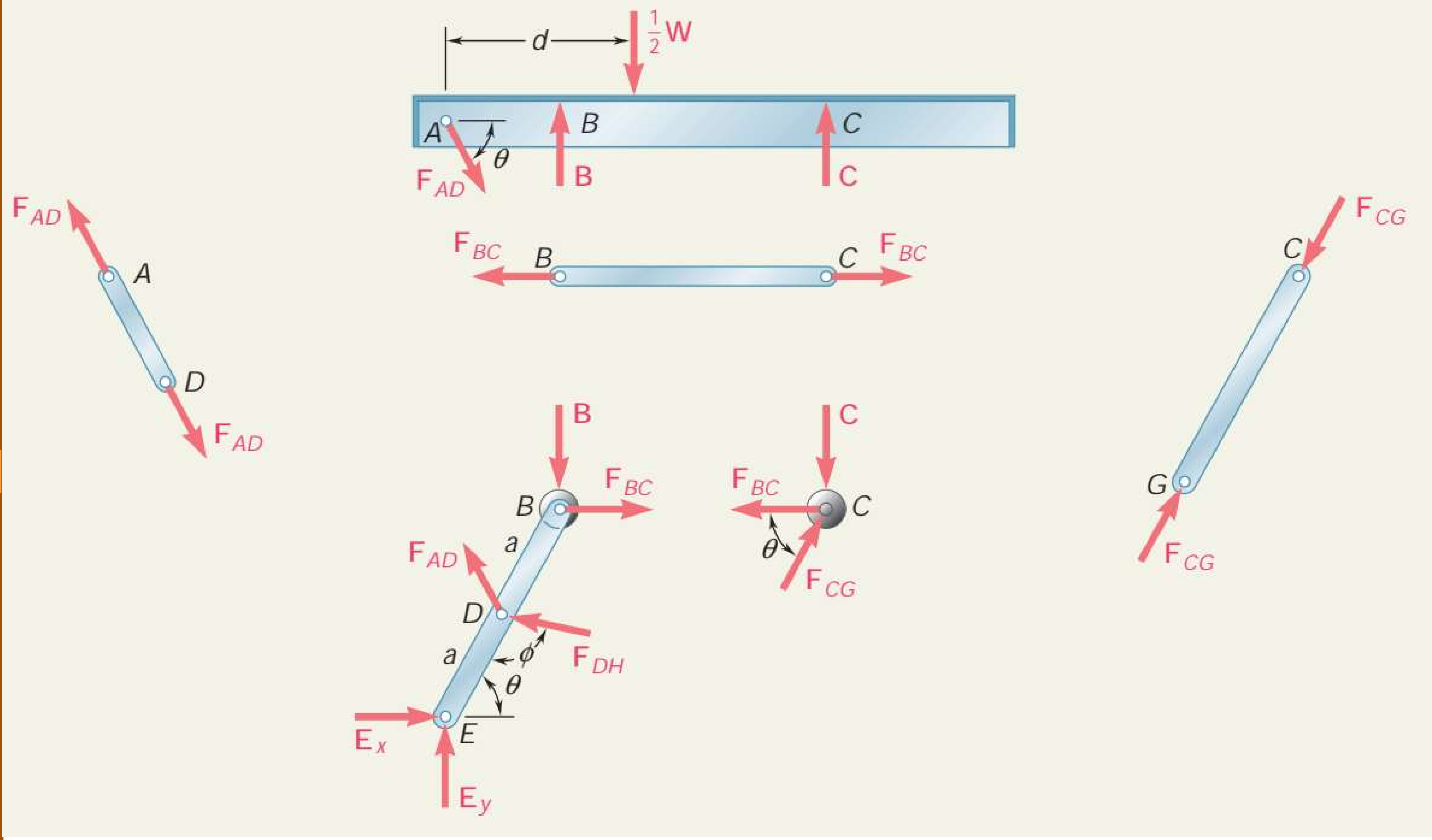
Sample Problem 6.7



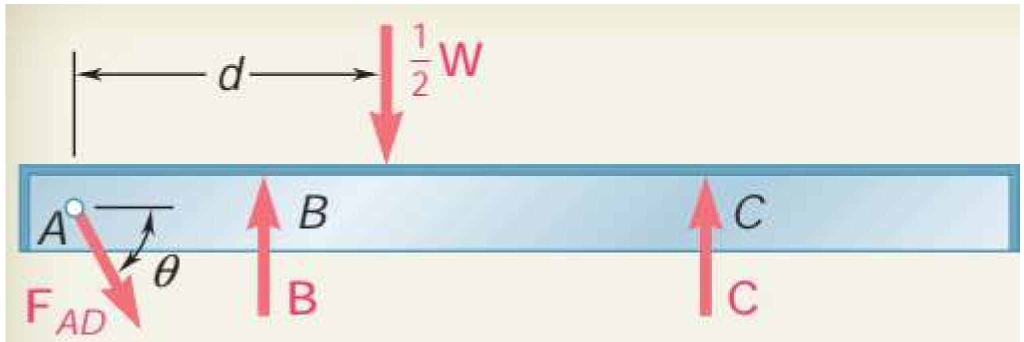
Sample Problem 6.7



Sample Problem 6.7



Sample Problem 6.7



Free Body: Platform *ABC*.

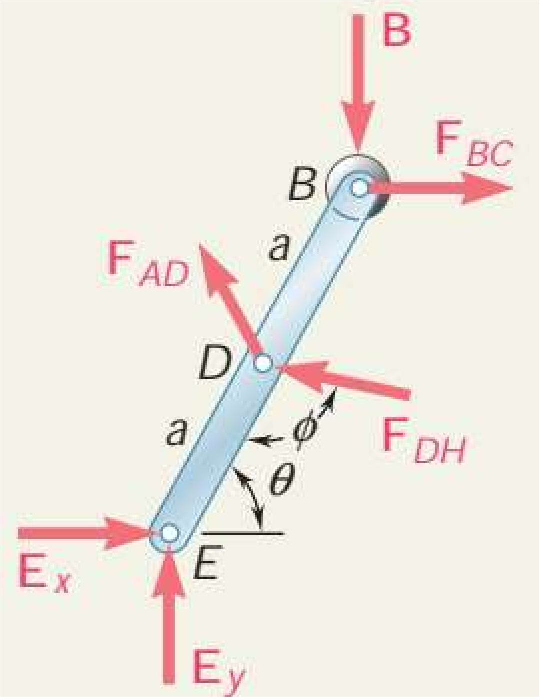
$\overset{+}{y}\ \Sigma F_x = 0$: $\quad F_{AD} \cos u = 0 \quad F_{AD} = 0$

$+x\ \Sigma F_y = 0$: $\quad B + C - \frac{1}{2}W = 0 \quad B + C = \frac{1}{2}W \quad (1)$

Sample Problem 6.7

Free Body: Roller C. We draw a force triangle and obtain $F_{BC} = C \cot \theta$.

Sample Problem 6.7



Sample Problem 6.7

Free Body: Member *BDE*. Recalling that $F_{AD} = 0$,

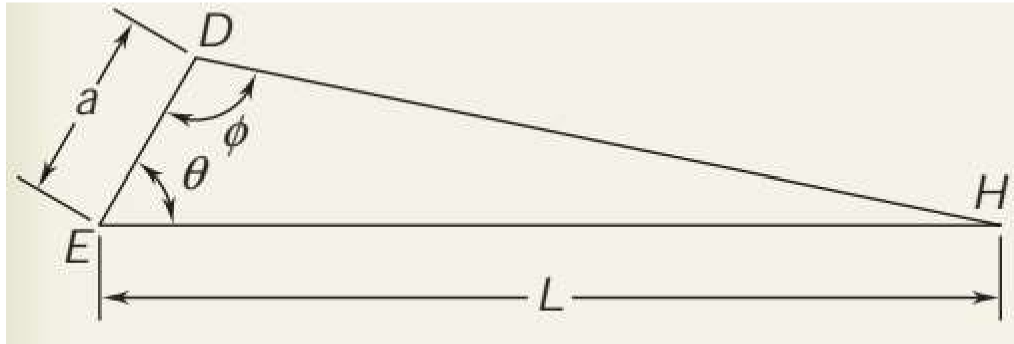
$$+l\Sigma M_E = 0: \qquad F_{DH} \cos (f - 90°)a - B(2a \cos u) - F_{BC}(2a \sin u) = 0$$
$$F_{DH} a \sin f - B(2a \cos u) - (C \cot u)(2a \sin u) = 0$$
$$F_{DH} \sin f - 2(B + C) \cos u = 0$$

Recalling Eq. (1), we have

$$F_{DH} = W\frac{\cos u}{\sin f} \qquad (2)$$

and we observe that *the result obtained is independent of d*.

Sample Problem 6.7



Applying first the law of sines to triangle EDH, we write

$$\frac{\sin \phi}{EH} = \frac{\sin \theta}{DH} \qquad \sin \phi = \frac{EH}{DH} \sin \theta$$

Using now the law of cosines, we have

$$\begin{aligned}(DH)^2 &= a^2 + L^2 - 2aL \cos \theta \\ &= (0.70)^2 + (3.20)^2 - 2(0.70)(3.20) \cos 60° \\ (DH)^2 &= 8.49 \qquad DH = 2.91 \text{ m}\end{aligned}$$

Sample Problem 6.7

We also note that

$$W = mg = (1000\ \text{kg})(9.81\ \text{m/s}^2) = 9810\ \text{N} = 9.81\ \text{kN}$$

Substituting for sin f from (3) into (2) and using the numerical data, we write

$$F_{DH} = W\frac{DH}{EH}\cot u = (9.81\ \text{kN})\frac{2.91\ \text{m}}{3.20\ \text{m}}\cot 60°$$

$$F_{DH} = 5.15\ \text{kN}$$

CHAPTER

7

VECTOR MECHANICS FOR ENGINEERS: STATICS

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Forces in Beams and Cables

Contents

Introduction

- Preceding chapters dealt with:

    a) determining external forces acting on a structure and

    b) determining forces which hold together the various members of a structure.

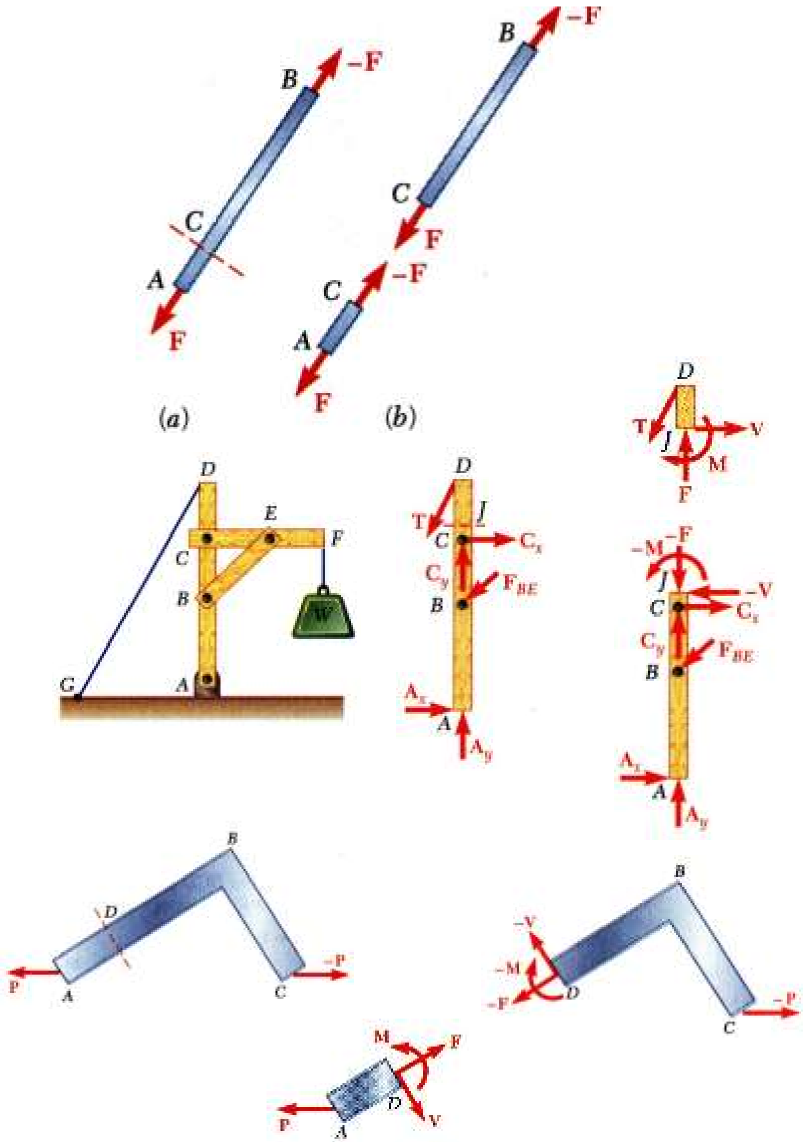
- The current chapter is concerned with determining the *internal forces* (i.e., tension/compression, shear, and bending) which hold together the various parts of a given member.

- Focus is on two important types of engineering structures:

    a) Beams - usually long, straight, prismatic members designed to support loads applied at various points along the member.

    b) Cables - flexible members capable of withstanding only tension, designed to support concentrated or distributed loads.

Internal Forces in Members



- Straight two-force member *AB* is in equilibrium under application of *F* and *-F*.

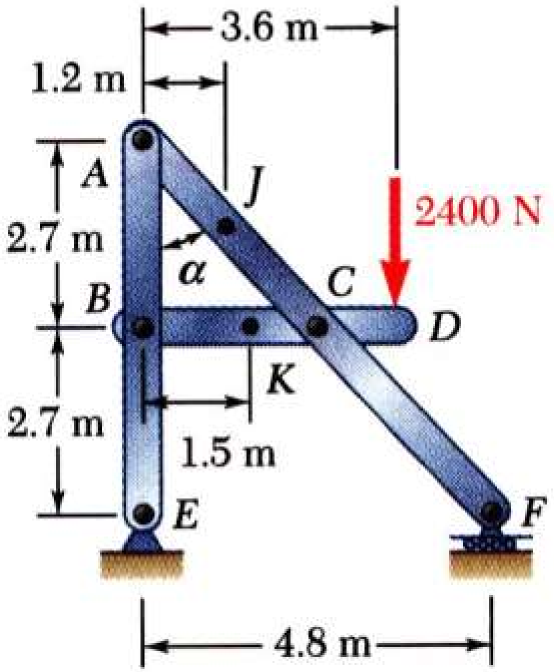
- *Internal forces* equivalent to *F* and *-F* are required for equilibrium of free-bodies *AC* and *CB*.

- Multiforce member *ABCD* is in equil-ibrium under application of cable and member contact forces.

- Internal forces equivalent to a force-couple system are necessary for equil-ibrium of free-bodies *JD* and *ABCJ*.

- An internal force-couple system is required for equilibrium of two-force members which are not straight.

Sample Problem 7.1

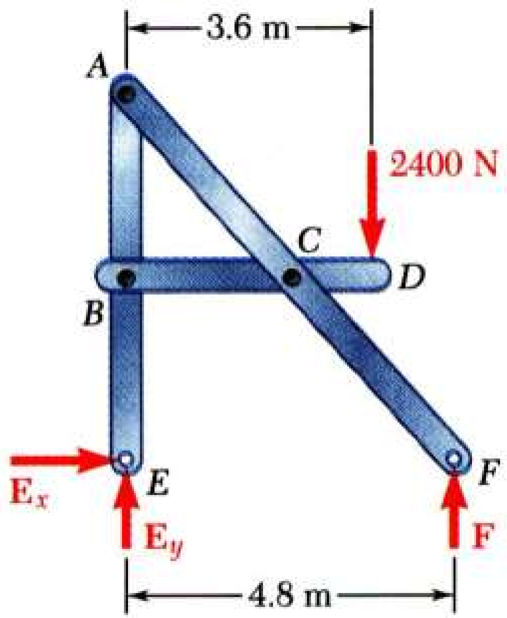


Determine the internal forces *(a)* in member *ACF* at point *J* and *(b)* in member *BCD* at *K*.

SOLUTION:

- Compute reactions and forces at connections for each member.
- Cut member *ACF* at *J*. The internal forces at *J* are represented by equivalent force-couple system which is determined by considering equilibrium of either part.
- Cut member *BCD* at *K*. Determine force-couple system equivalent to internal forces at *K* by applying equilibrium conditions to either part.

Sample Problem 7.1



SOLUTION:

- Compute reactions and connection forces.

Consider entire frame as a free-body:

$$\sum M_E = 0:$$

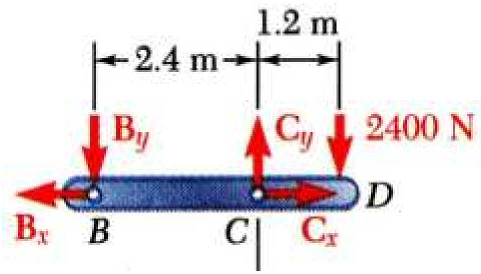
$$-(2400\,\text{N})(3.6\,\text{m}) + F(4.8\,\text{m}) = 0 \qquad F = 1800\,\text{N}$$

$$\sum F_y = 0:$$

$$-2400\,\text{N} + 1800\,\text{N} + E_y = 0 \qquad E_y = 600\,N$$

$$\sum F_x = 0: \qquad E_x = 0$$

Sample Problem 7.1



Consider member *BCD* as free-body:

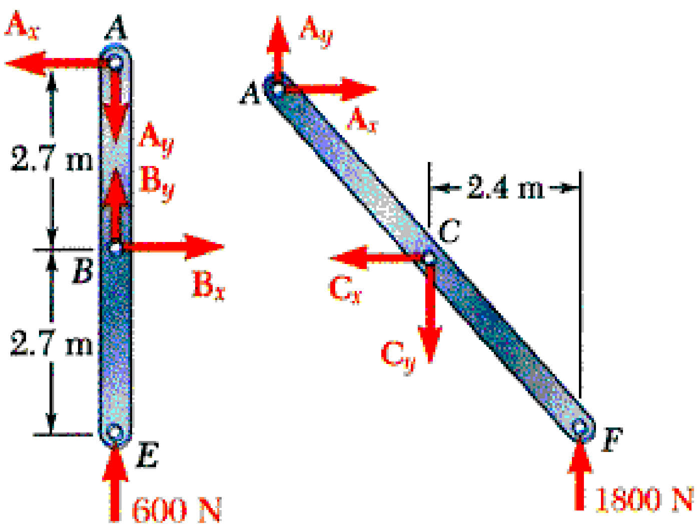
$$\sum M_B = 0:$$

$$-(2400\,\text{N})(3.6\,\text{m}) + C_y(2.4\,\text{m}) = 0 \qquad C_y = 3600\,\text{N}$$

$$\sum M_C = 0:$$

$$-(2400\,\text{N})(1.2\,\text{m}) + B_y(2.4\,\text{m}) = 0 \qquad B_y = 1200\,\text{N}$$

$$\sum F_x = 0: \qquad -B_x + C_x = 0$$



Consider member *ABE* as free-body:

$$\sum M_A = 0: \qquad B_x(2.4\,\text{m}) = 0 \qquad B_x = 0$$

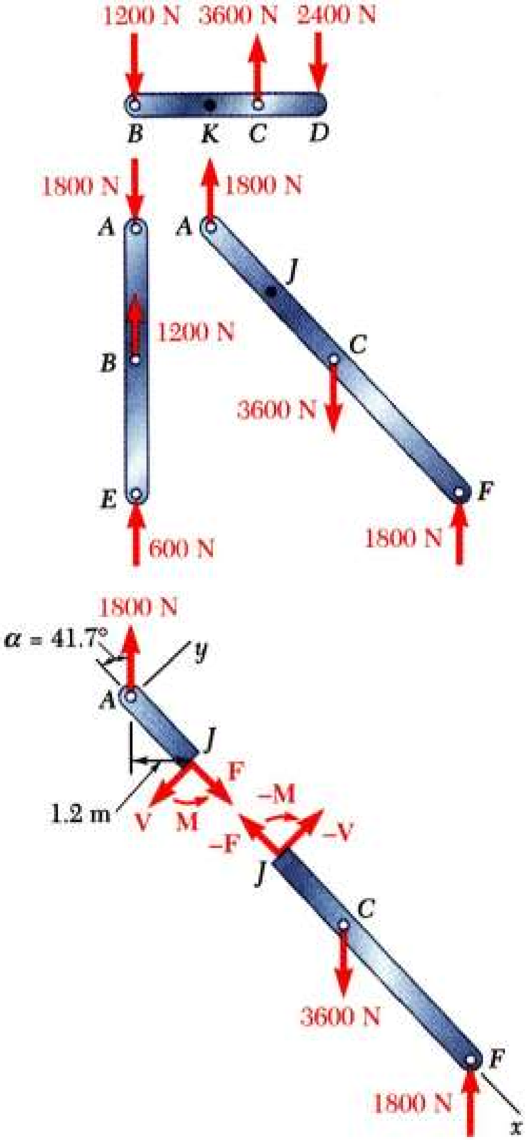
$$\sum F_x = 0: \qquad B_x - A_x = 0 \qquad A_x = 0$$

$$\sum F_y = 0: \qquad -A_y + B_y + 600\,\text{N} = 0 \qquad A_y = 1800\,\text{N}$$

From member *BCD*,

$$\sum F_x = 0: \qquad -B_x + C_x = 0 \qquad C_x = 0$$

Sample Problem 7.1



- Cut member ACF at J. The internal forces at J are represented by equivalent force-couple system.

Consider free-body AJ:

$$\sum M_J = 0:$$

$$-(1800\,\mathrm{N})(1.2\,\mathrm{m}) + M = 0 \qquad \boxed{M = 2160\,\mathrm{N\cdot m}}$$

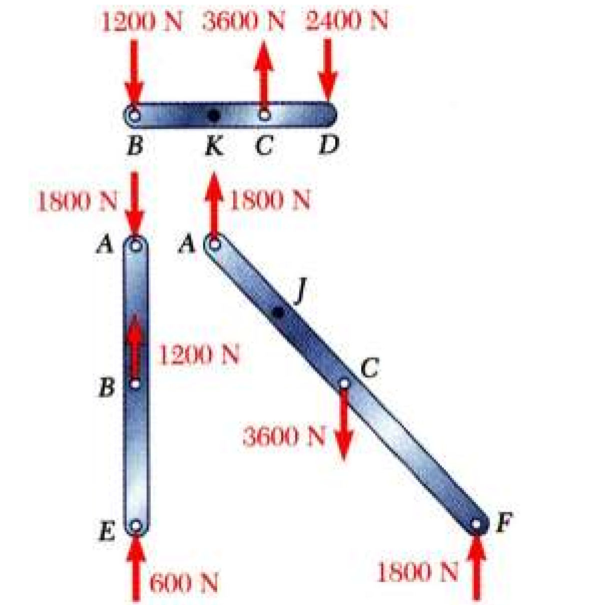
$$\sum F_x = 0:$$

$$F - (1800\,\mathrm{N})\cos 41.7° = 0 \qquad \boxed{F = 1344\,\mathrm{N}}$$

$$\sum F_y = 0:$$

$$-V + (1800\,\mathrm{N})\sin 41.7° = 0 \qquad \boxed{V = 1197\,\mathrm{N}}$$

Sample Problem 7.1



- Cut member *BCD* at *K*. Determine a force-couple system equivalent to internal forces at *K* .

Consider free-body *BK*:

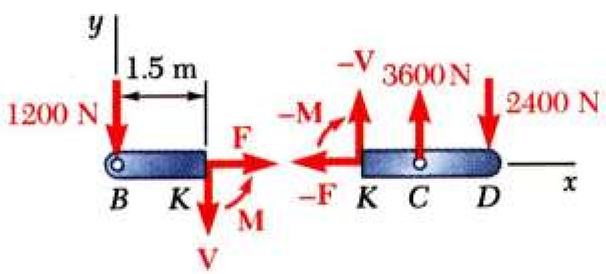
$$\sum M_K = 0:$$

$$(1200\,\mathrm{N})(1.5\,\mathrm{m}) + M = 0 \qquad \boxed{M = -1800\,\mathrm{N\cdot m}}$$

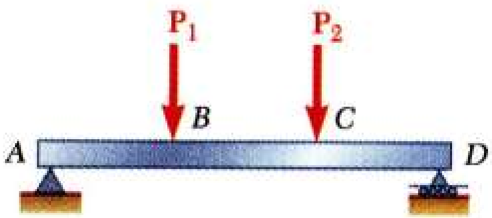
$$\sum F_x = 0: \qquad \boxed{F = 0}$$

$$\sum F_y = 0:$$

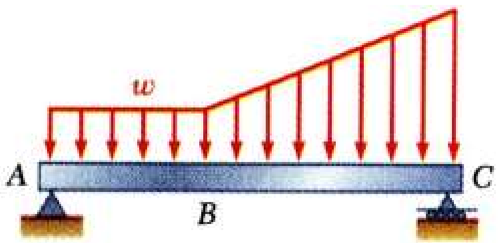
$$-1200\,\mathrm{N} - V = 0 \qquad \boxed{V = -1200\,\mathrm{N}}$$



Various Types of Beam Loading and Support



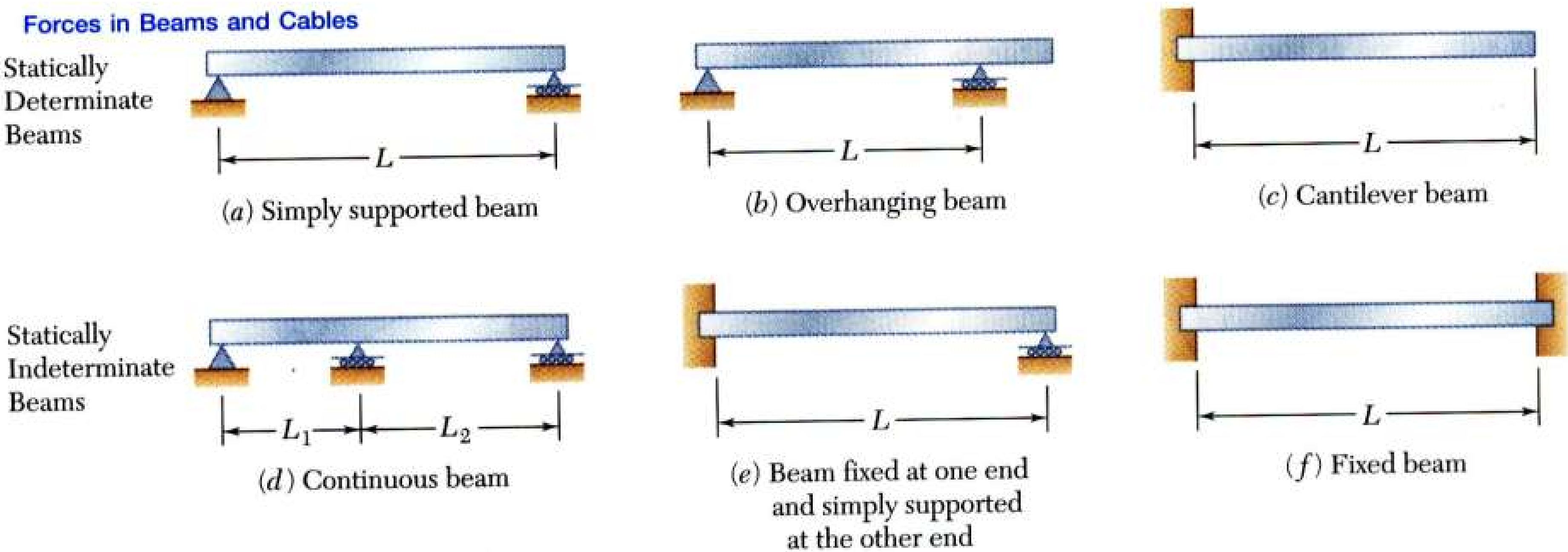
(*a*) Concentrated loads



(*b*) Distributed load

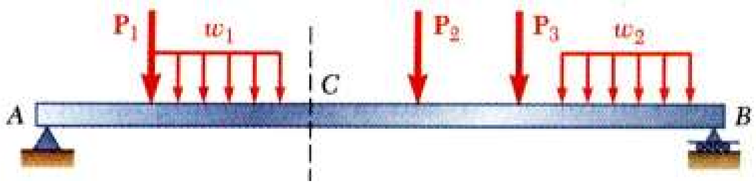
- *Beam* - structural member designed to support loads applied at various points along its length.
- Beam can be subjected to *concentrated* loads or *distributed* loads or combination of both.
- *Beam design* is two-step process:
  1) determine shearing forces and bending moments produced by applied loads
  2) select cross-section best suited to resist shearing forces and bending moments

Various Types of Beam Loading and Support

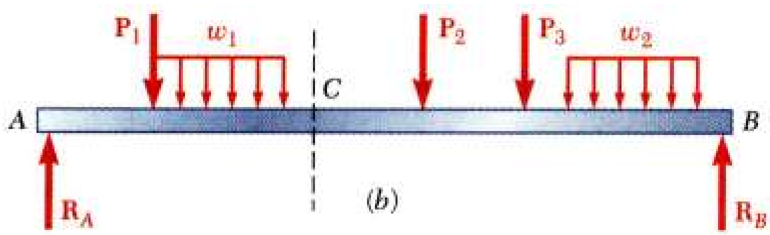


- Beams are classified according to way in which they are supported.
- Reactions at beam supports are determinate if they involve only three unknowns. Otherwise, they are statically indeterminate.

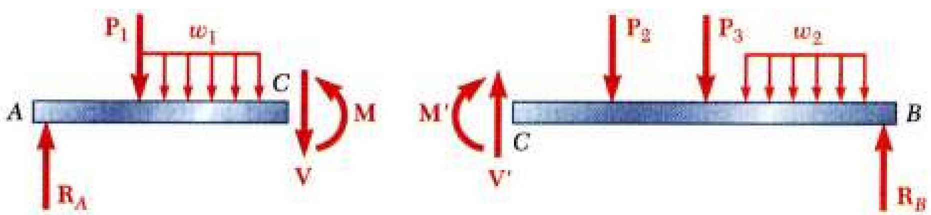
Shear and Bending Moment in a Beam



- Wish to determine bending moment and shearing force at any point in a beam subjected to concentrated and distributed loads.



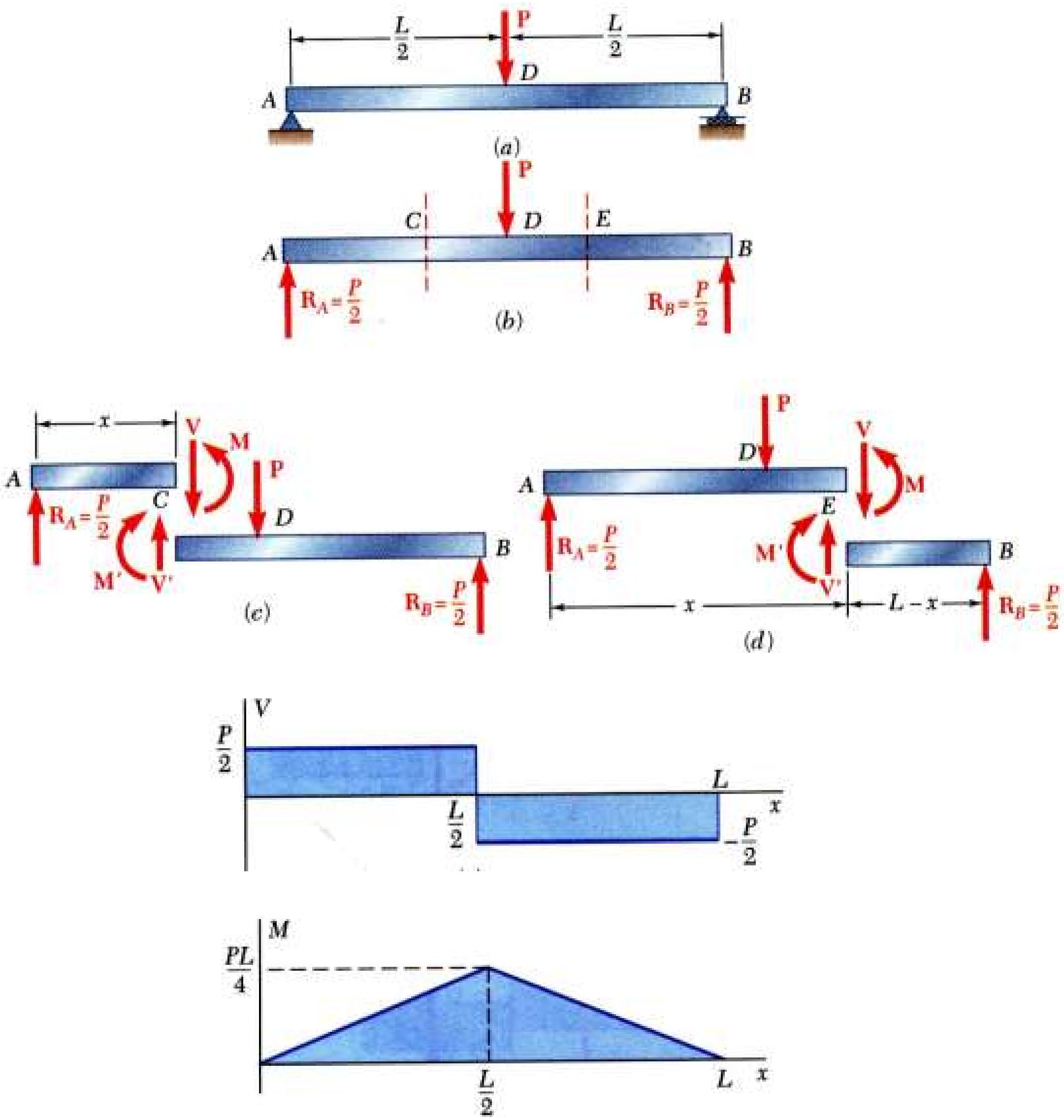
- Determine reactions at supports by treating whole beam as free-body.



- Cut beam at *C* and draw free-body diagrams for *AC* and *CB*. By definition, positive sense for internal force-couple systems are as shown.

- From equilibrium considerations, determine *M* and *V* or *M'* and *V'*.

Shear and Bending Moment Diagrams

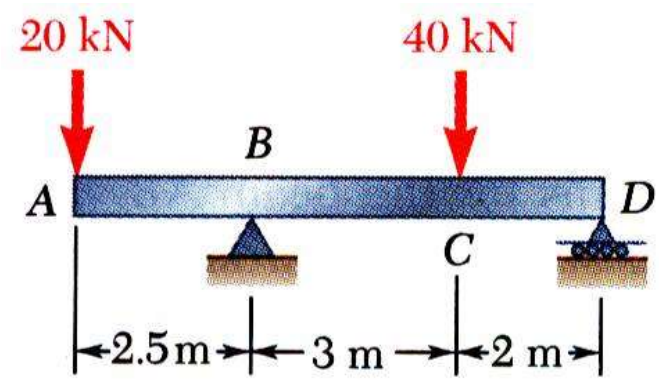


- Variation of shear and bending moment along beam may be plotted.
- Determine reactions at supports.
- Cut beam at C and consider member AC,

  $V = +P/2 \quad M = +Px/2$
- Cut beam at E and consider member EB,

  $V = -P/2 \quad M = +P(L-x)/2$
- For a beam subjected to concentrated loads, shear is constant between loading points and moment varies linearly.

Sample Problem 7.2

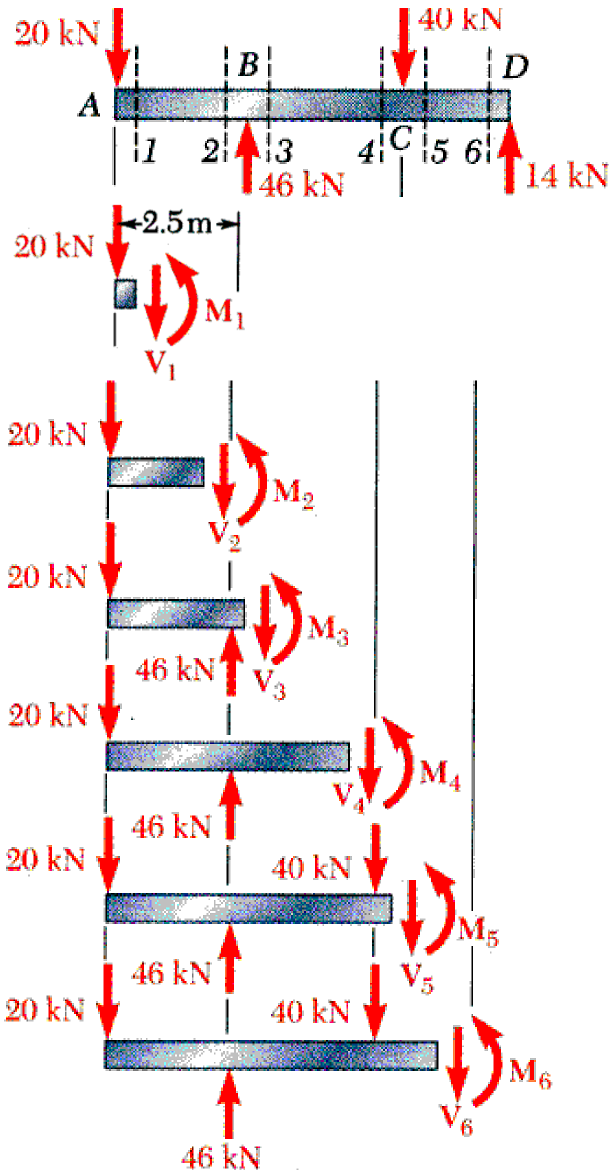


Draw the shear and bending moment diagrams for the beam and loading shown.

SOLUTION:

- Taking entire beam as a free-body, calculate reactions at *B* and *D*.
- Find equivalent internal force-couple systems for free-bodies formed by cutting beam on either side of load application points.
- Plot results.

Sample Problem 7.2



SOLUTION:

- Taking entire beam as a free-body, calculate reactions at *B* and *D*.
- Find equivalent internal force-couple systems at sections on either side of load application points.

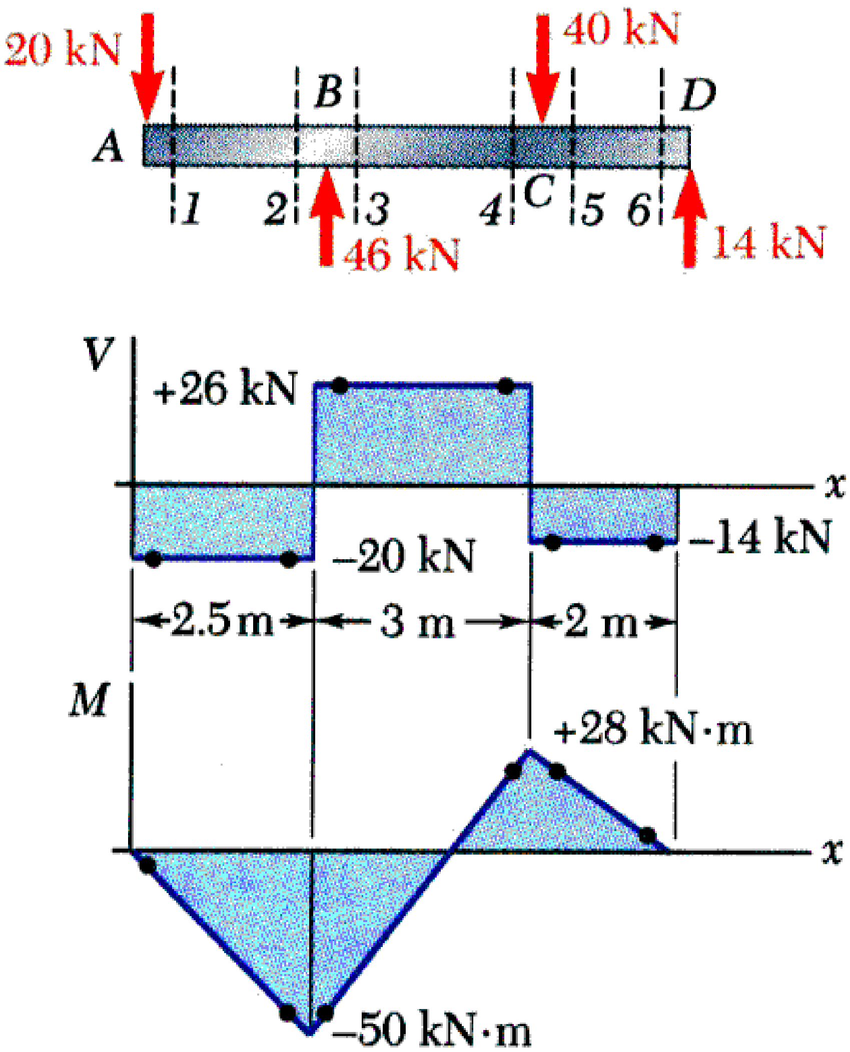
$$\sum F_y = 0: \quad -20\,\mathrm{kN} - V_1 = 0 \qquad \boxed{V_1 = -20\,\mathrm{kN}}$$

$$\sum M_2 = 0: \quad (20\,\mathrm{kN})(0\,\mathrm{m}) + M_1 = 0 \qquad \boxed{M_1 = 0}$$

Similarly,

$$\boxed{\begin{array}{ll} V_3 = 26\,\mathrm{kN} & M_3 = -50\,\mathrm{kN\cdot m} \\ V_4 = 26\,\mathrm{kN} & M_4 = -50\,\mathrm{kN\cdot m} \\ V_5 = 26\,\mathrm{kN} & M_5 = -50\,\mathrm{kN\cdot m} \\ V_6 = 26\,\mathrm{kN} & M_6 = -50\,\mathrm{kN\cdot m} \end{array}}$$

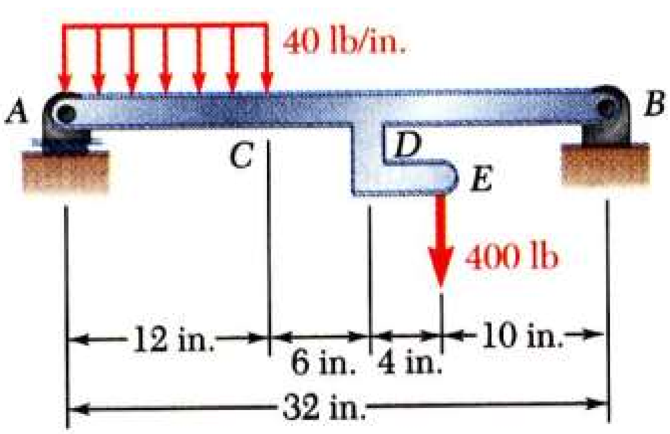
Sample Problem 7.2



- Plot results.

  Note that shear is of constant value between concentrated loads and bending moment varies linearly.

Sample Problem 7.3

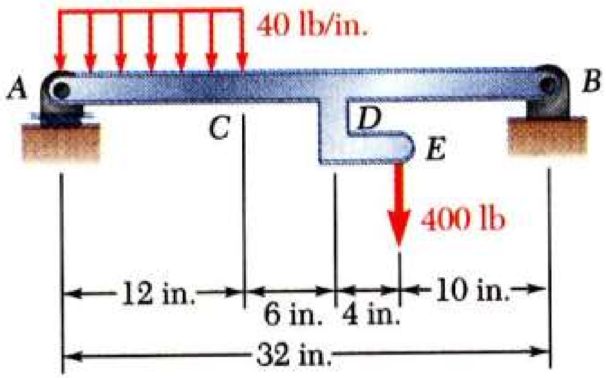


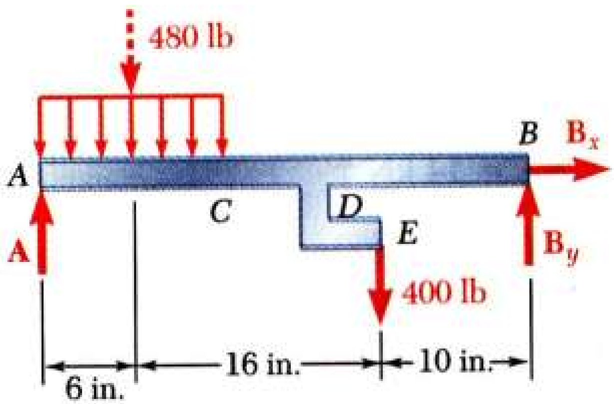
Draw the shear and bending moment diagrams for the beam *AB*. The distributed load of 40 lb/in. extends over 12 in. of the beam, from *A* to *C*, and the 400 lb load is applied at *E*.

SOLUTION:

- Taking entire beam as free-body, calculate reactions at *A* and *B*.
- Determine equivalent internal force-couple systems at sections cut within segments *AC*, *CD*, and *DB*.
- Plot results.

Sample Problem 7.3





SOLUTION:

- Taking entire beam as a free-body, calculate reactions at *A* and *B*.

$$\sum M_A = 0:$$

$$B_y(32\,\text{in.}) - (480\,\text{lb})(6\,\text{in.}) - (400\,\text{lb})(22\,\text{in.}) = 0$$

$$\boxed{B_y = 365\,\text{lb}}$$

$$\sum M_B = 0:$$

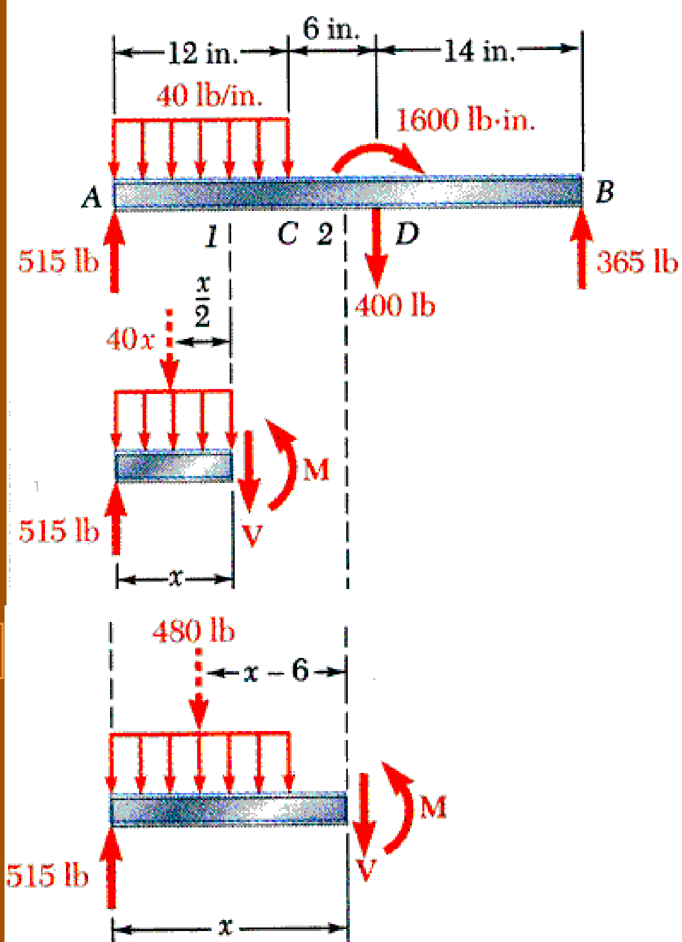
$$(480\,\text{lb})(26\,\text{in.}) + (400\,\text{lb})(10\,\text{in.}) - A(32\,\text{in.}) = 0$$

$$\boxed{A = 515\,\text{lb}}$$

$$\sum F_x = 0: \qquad \boxed{B_x = 0}$$

- Note: The 400 lb load at *E* may be replaced by a 400 lb force and 1600 lb-in. couple at *D*.

Sample Problem 7.3



- Evaluate equivalent internal force-couple systems at sections cut within segments *AC*, *CD*, and *DB*.

From A to *C*:

$$\sum F_y = 0: \quad 515 - 40x - V = 0$$

$$\boxed{V = 515 - 40x}$$

$$\sum M_1 = 0: \quad -515x - 40x\left(\tfrac{1}{2}x\right) + M = 0$$

$$\boxed{M = 515x - 20x^2}$$

From *C* to *D*:

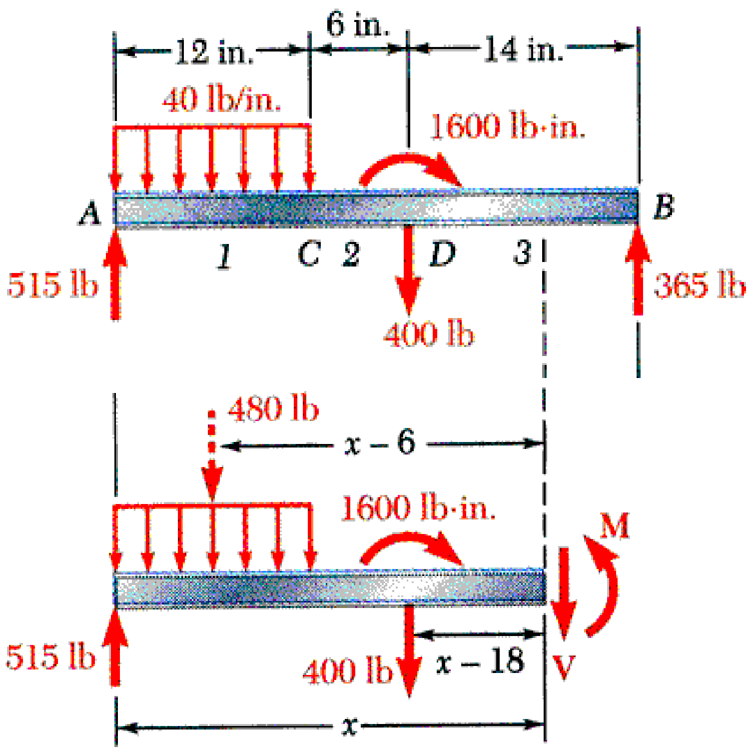
$$\sum F_y = 0: \quad 515 - 480 - V = 0$$

$$\boxed{V = 35\,\text{lb}}$$

$$\sum M_2 = 0: \quad -515x + 480(x - 6) + M = 0$$

$$\boxed{M = (2880 + 35x)\,\text{lb}\cdot\text{in.}}$$

Sample Problem 7.3



- Evaluate equivalent internal force-couple systems at sections cut within segments *AC*, *CD*, and *DB*.

From *D* to *B*:

$$\sum F_y = 0: \qquad 515 - 480 - 400 - V = 0$$

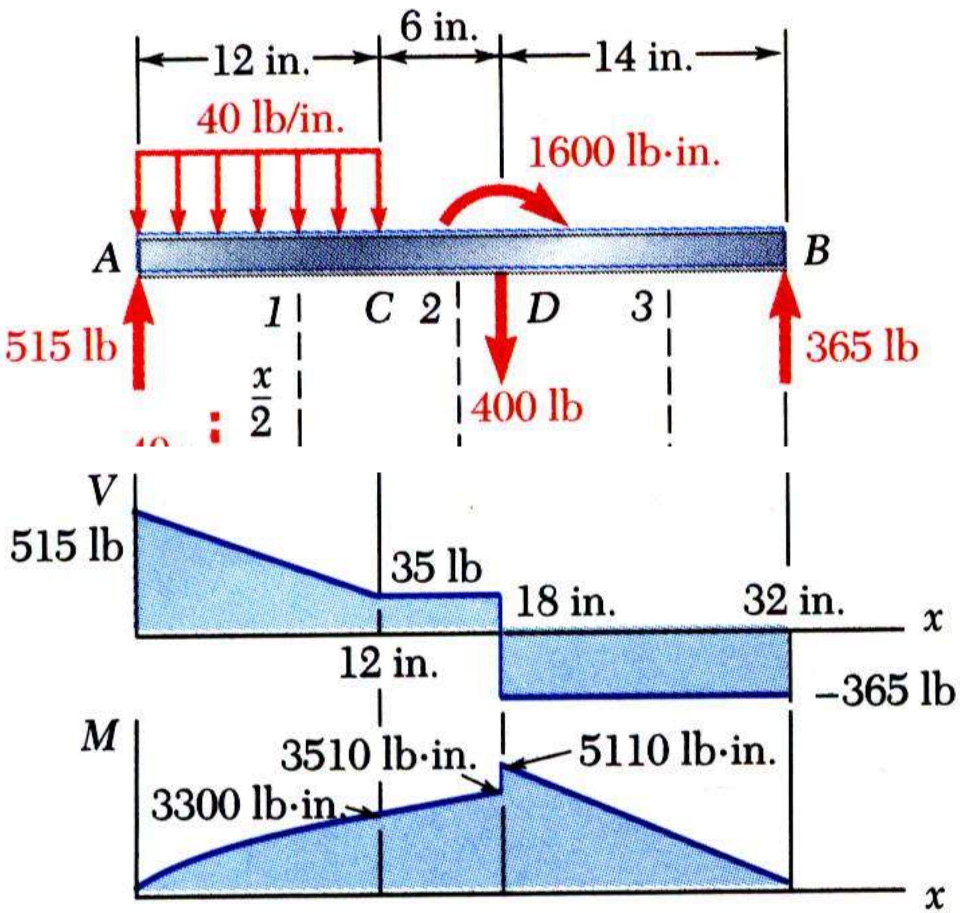
$$\boxed{V = -365\,\text{lb}}$$

$$\sum M_2 = 0:$$

$$-515x + 480(x-6) - 1600 + 400(x-18) + M = 0$$

$$\boxed{M = (11{,}680 - 365x)\,\text{lb}\cdot\text{in.}}$$

Sample Problem 7.3



- Plot results.

From A to C:

$$\boxed{V = 515 - 40x}$$

$$\boxed{M = 515x - 20x^2}$$

From C to D:

$$\boxed{V = 35\,\text{lb}}$$

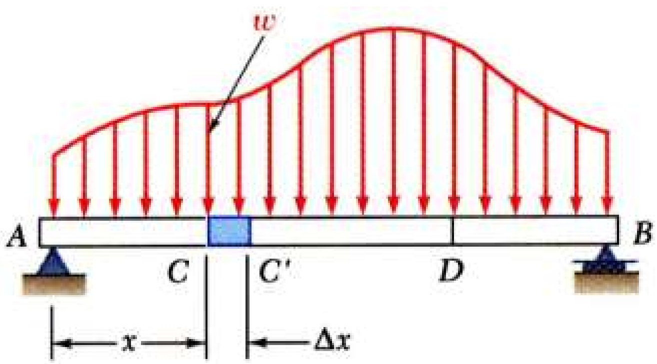
$$\boxed{M = (2880 + 35x)\,\text{lb}\cdot\text{in.}}$$

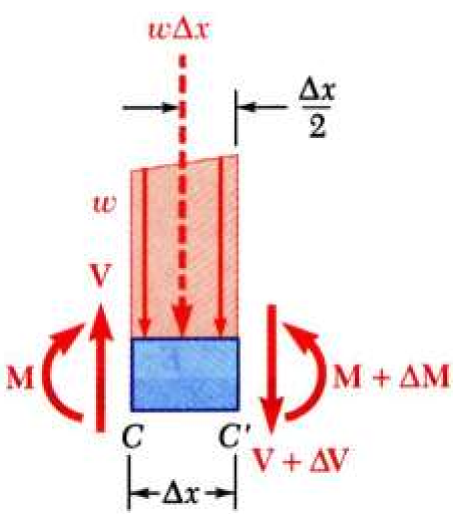
From D to B:

$$\boxed{V = -365\,\text{lb}}$$

$$\boxed{M = (11{,}680 - 365x)\,\text{lb}\cdot\text{in.}}$$

Relations Among Load, Shear, and Bending Moment





- Relations between load and shear:

$$V-(V+\Delta V)-w\Delta x=0$$

$$\frac{dV}{dx}=\lim_{\Delta x\to 0}\frac{\Delta V}{\Delta x}=-w$$

$$V_D-V_C=-\int_{x_C}^{x_D} w\,dx=-(\text{area under load curve})$$

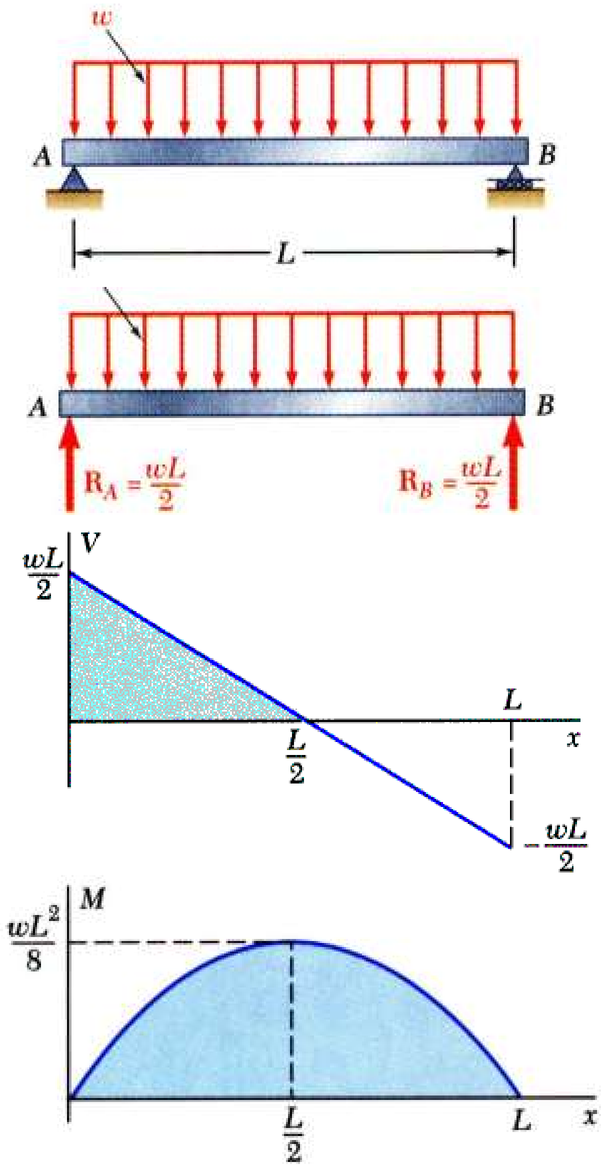
- Relations between shear and bending moment:

$$(M+\Delta M)-M-V\Delta x+w\Delta x\frac{\Delta x}{2}=0$$

$$\frac{dM}{dx}=\lim_{\Delta x\to 0}\frac{\Delta M}{\Delta x}=\lim_{\Delta x\to 0}\left(V-\tfrac{1}{2}w\Delta x\right)=V$$

$$M_D-M_C=\int_{x_C}^{x_D} V\,dx=(\text{area under shear curve})$$

Relations Among Load, Shear, and Bending Moment



- Reactions at supports, $R_A = R_B = \dfrac{wL}{2}$

- Shear curve,

$$V - V_A = -\int_0^x w\,dx = -wx$$

$$V = V_A - wx = \frac{wL}{2} - wx = w\left(\frac{L}{2} - x\right)$$

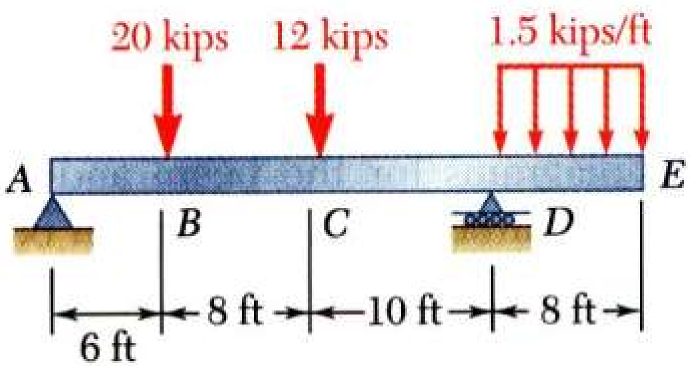
- Moment curve,

$$M - M_A = \int_0^x V dx$$

$$M = \int_0^x w\left(\frac{L}{2} - x\right)dx = \frac{w}{2}\left(Lx - x^2\right)$$

$$M_{max} = \frac{wL^2}{8} \quad \left(M \text{ at } \frac{dM}{dx} = V = 0\right)$$

Sample Problem 7.4

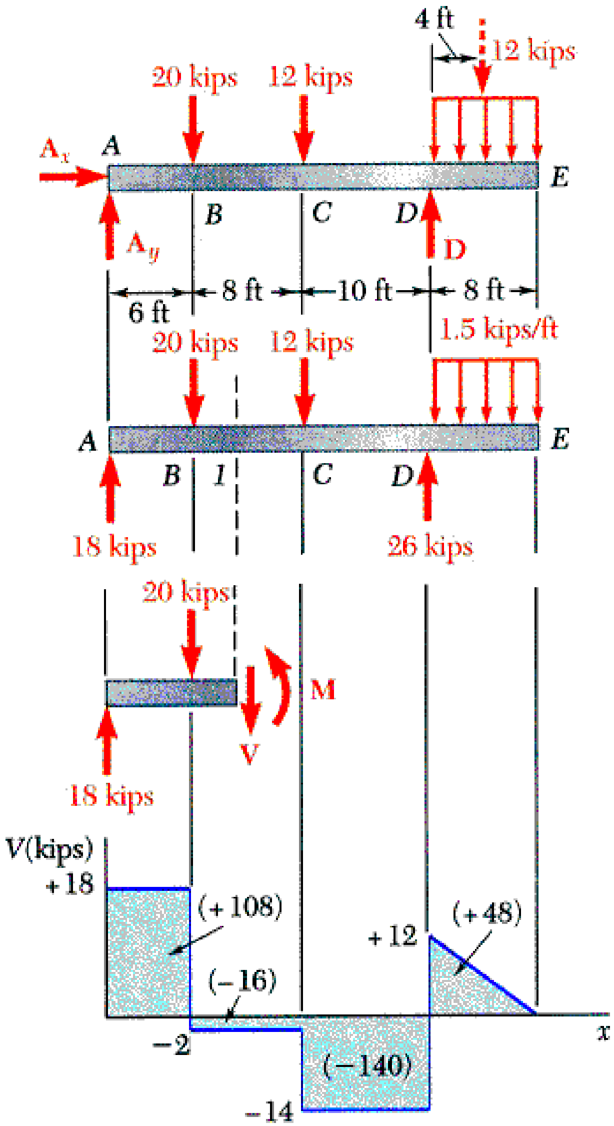


Draw the shear and bending-moment diagrams for the beam and loading shown.

SOLUTION:

- Taking entire beam as a free-body, determine reactions at supports.
- Between concentrated load application points, $dV/dx = -w = 0$ and shear is constant.
- With uniform loading between D and E, the shear variation is linear.
- Between concentrated load application points, $dM/dx = V = \text{constant}$. The change in moment between load application points is equal to area under shear curve between points.
- With a linear shear variation between D and E, the bending moment diagram is a parabola.

Sample Problem 7.4



SOLUTION:

- Taking entire beam as a free-body, determine reactions at supports.

$$\sum M_A = 0:$$

$$D(24\text{ ft}) - (20\text{ kips})(6\text{ ft}) - (12\text{ kips})(14\text{ ft}) - (12\text{ kips})(28\text{ ft}) = 0$$

$$\boxed{D = 26\text{ kips}}$$

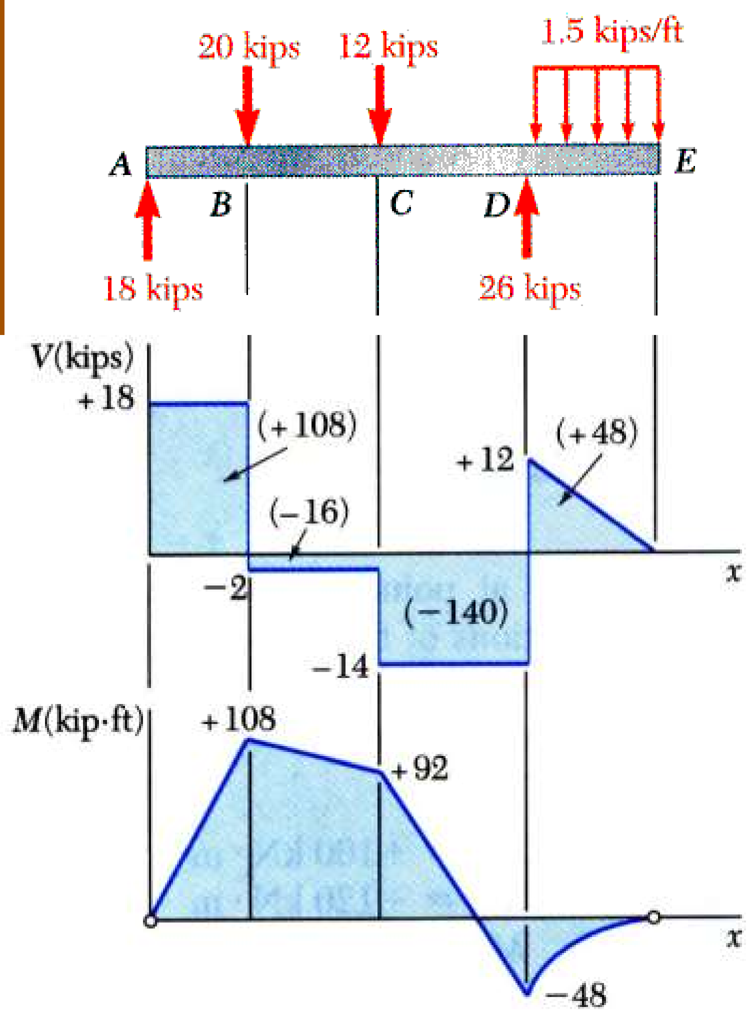
$$\sum F_y = 0:$$

$$A_y - 20\text{ kips} - 12\text{ kips} + 26\text{ kips} - 12\text{ kips} = 0$$

$$\boxed{A_y = 18\text{ kips}}$$

- Between concentrated load application points, $dV/dx = -w = 0$ and shear is constant.
- With uniform loading between D and E, the shear variation is linear.

Sample Problem 7.4



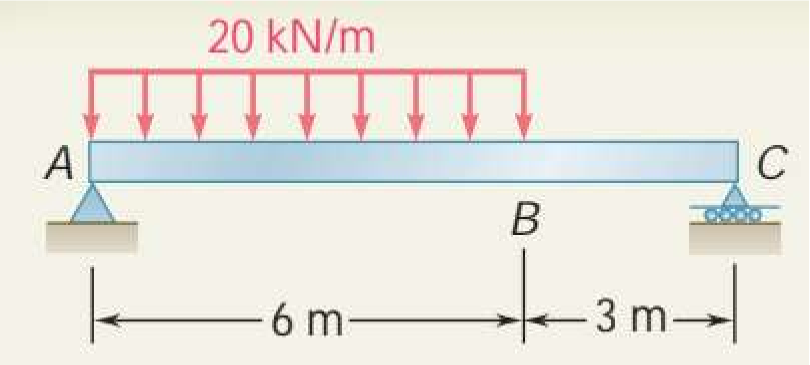
- Between concentrated load application points, $dM/dx = V = \text{constant}$. The change in moment between load application points is equal to area under the shear curve between points.

$$
\begin{aligned}
M_B - M_A &= +108 & M_B &= +108\,\text{kip}\cdot\text{ft} \\
M_C - M_B &= -16 & M_C &= +92\,\text{kip}\cdot\text{ft} \\
M_D - M_C &= -140 & M_D &= -48\,\text{kip}\cdot\text{ft} \\
M_E - M_D &= +48 & M_E &= 0
\end{aligned}
$$

- With a linear shear variation between D and E, the bending moment diagram is a parabola.

Sample Problem 7.5

Draw the shear and bending-moment diagrams for the beam and loading shown and determine the location and magnitude of the maximum bending moment.



Sample Problem 7.5

Free-Body: Entire Beam. Considering the entire beam as a free body, we obtain the reactions

$$\mathbf{R}_A = 80 \text{ kNx} \qquad \mathbf{R}_C = 40 \text{ kNx}$$

Shear Diagram. The shear just to the right of A is $V_A = +80$ kN. Since the change in shear between two points is equal to *minus* the area under the load curve between the same two points, we obtain V_B by writing

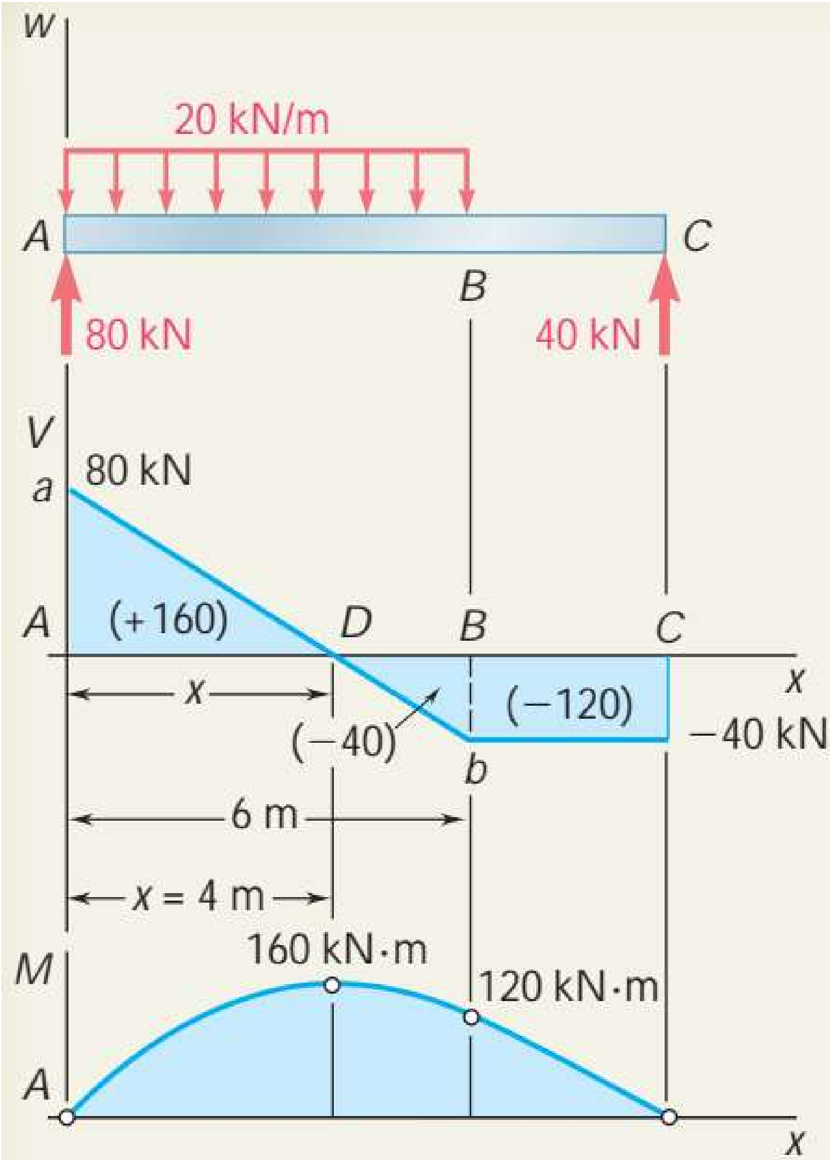
$$V_B - V_A = -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN}$$
$$V_B = -120 + V_A = -120 + 80 = -40 \text{ kN}$$

Since the slope $dV/dx = -w$ is constant between A and B, the shear diagram between these two points is represented by a straight line. Between B and C, the area under the load curve is zero; therefore,

$$V_C - V_B = 0 \qquad V_C = V_B = -40 \text{ kN}$$

and the shear is constant between B and C.

Sample Problem 7.5



Sample Problem 7.5

Bending-Moment Diagram. We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section D of the beam where $V = 0$. We write

$$V_D - V_A = -wx$$
$$0 - 80 \text{ kN} = -(20 \text{ kN/m})x$$

and, solving for x: $x = 4 \text{ m}$ ◀

The maximum bending moment occurs at point D, where we have $dM/dx = V = 0$. The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

$$M_D - M_A = +160 \text{ kN} \cdot \text{m} \qquad M_D = +160 \text{ kN} \cdot \text{m}$$
$$M_B - M_D = -\ 40 \text{ kN} \cdot \text{m} \qquad M_B = +120 \text{ kN} \cdot \text{m}$$
$$M_C - M_B = -120 \text{ kN} \cdot \text{m} \qquad M_C = 0$$

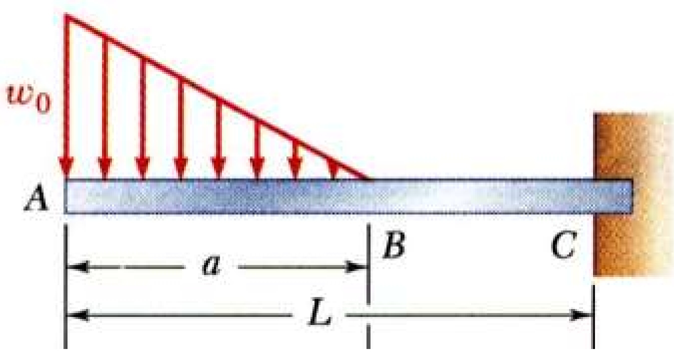
Sample Problem 7.5

The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at A is equal to the value of V at that point.

The maximum bending moment is

$$M_{max} = M_D = +160 \text{ kN} \cdot \text{m}$$

Sample Problem 7.6

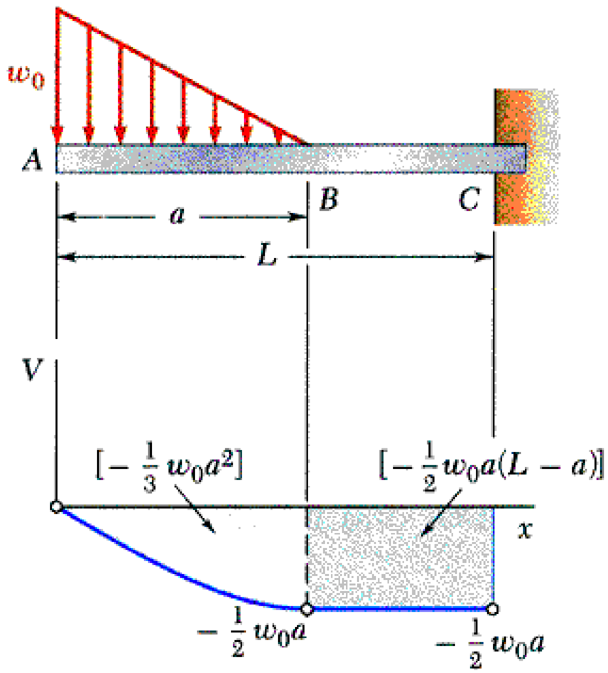


Sketch the shear and bending-moment diagrams for the cantilever beam and loading shown.

SOLUTION:

- The change in shear between *A* and *B* is equal to the negative of area under load curve between points. The linear load curve results in a parabolic shear curve.
- With zero load, change in shear between *B* and *C* is zero.
- The change in moment between *A* and *B* is equal to area under shear curve between points. The parabolic shear curve results in a cubic moment curve.
- The change in moment between *B* and *C* is equal to area under shear curve between points. The constant shear curve results in a linear moment curve.

Sample Problem 7.6



SOLUTION:

- The change in shear between A and B is equal to negative of area under load curve between points. The linear load curve results in a parabolic shear curve.

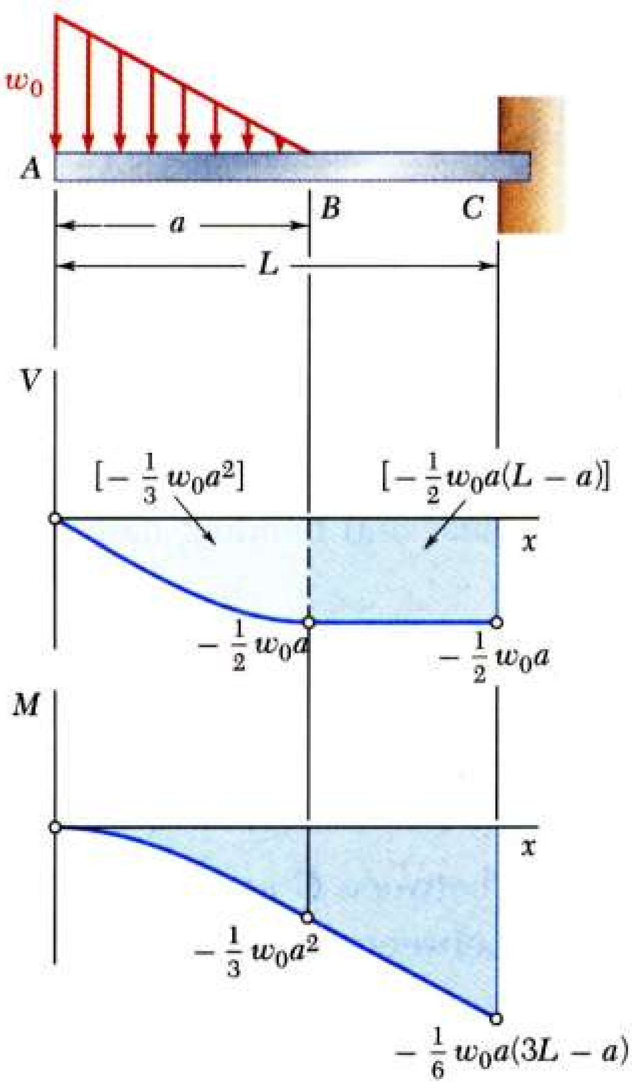
$$\text{at } A, \quad V_A = 0, \quad \frac{dV}{dx} = -w = -w_0$$

$$V_B - V_A = -\tfrac{1}{2}w_0 a \qquad\qquad V_B = -\tfrac{1}{2}w_0 a$$

$$\text{at } B, \quad \frac{dV}{dx} = -w = 0$$

- With zero load, change in shear between B and C is zero.

Sample Problem 7.6



- The change in moment between *A* and *B* is equal to area under shear curve between the points. The parabolic shear curve results in a cubic moment curve.

$$\text{at } A,\quad M_A = 0,\quad \frac{dM}{dx} = V = 0$$

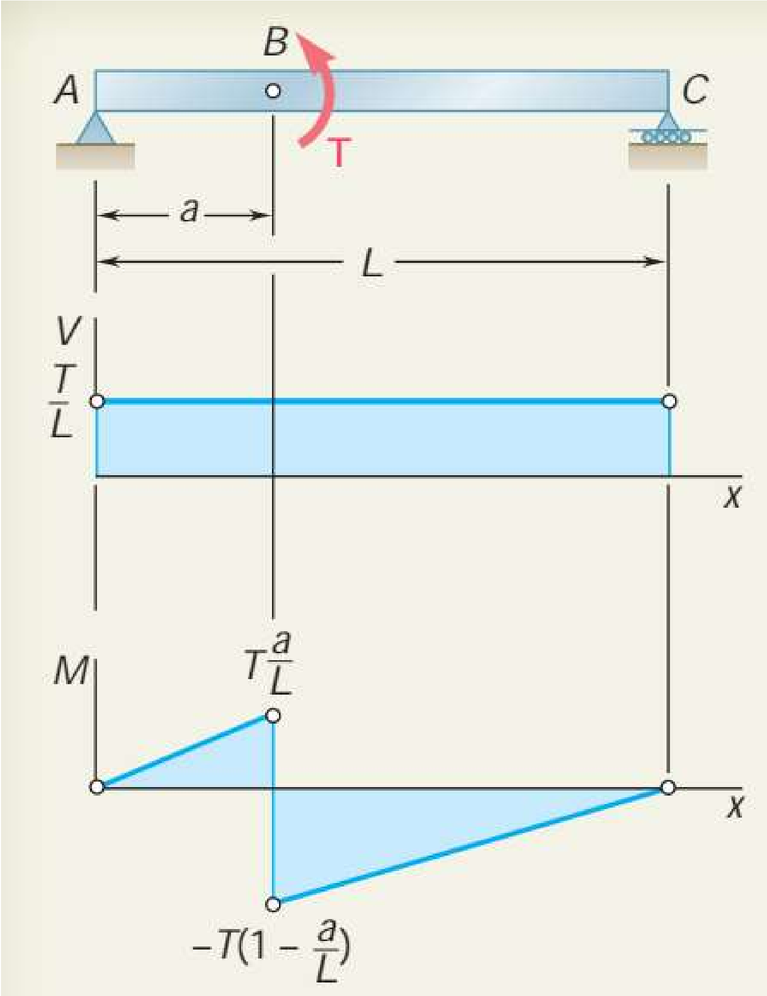
$$M_B - M_A = -\tfrac{1}{3}w_0a^2 \qquad M_B = -\tfrac{1}{3}w_0a^2$$

$$M_C - M_B = -\tfrac{1}{2}w_0a(L-a) \qquad M_C = -\tfrac{1}{6}w_0a(3L-a)$$

- The change in moment between *B* and *C* is equal to area under shear curve between points. The constant shear curve results in a linear moment curve.

Sample Problem 7.7

The simple beam AC is loaded by a couple of magnitude T applied at point B. Draw the shear and bending-moment diagrams for the beam.



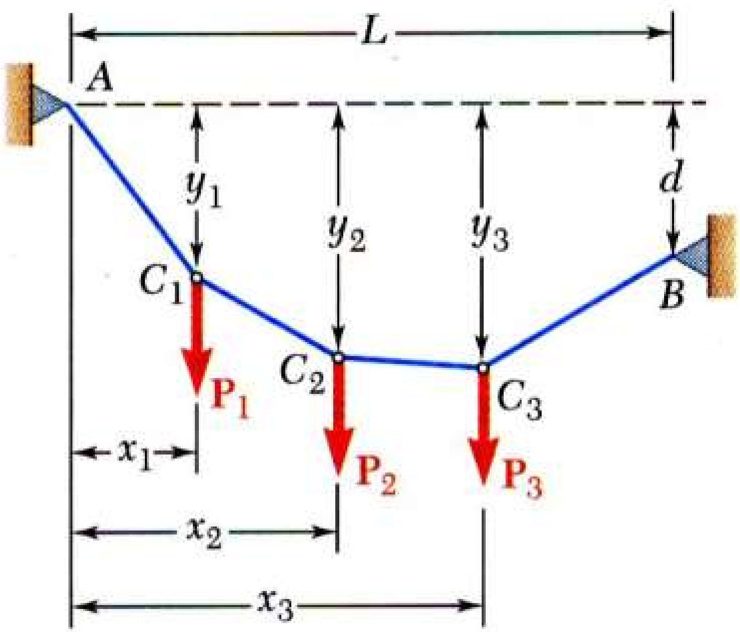
Sample Problem 7.7

Free-Body: Entire Beam. The entire beam is taken as a free body, and we obtain

$$\mathbf{R}_A = \frac{T}{L}\text{x} \qquad \mathbf{R}_C = \frac{T}{L}\text{w}$$

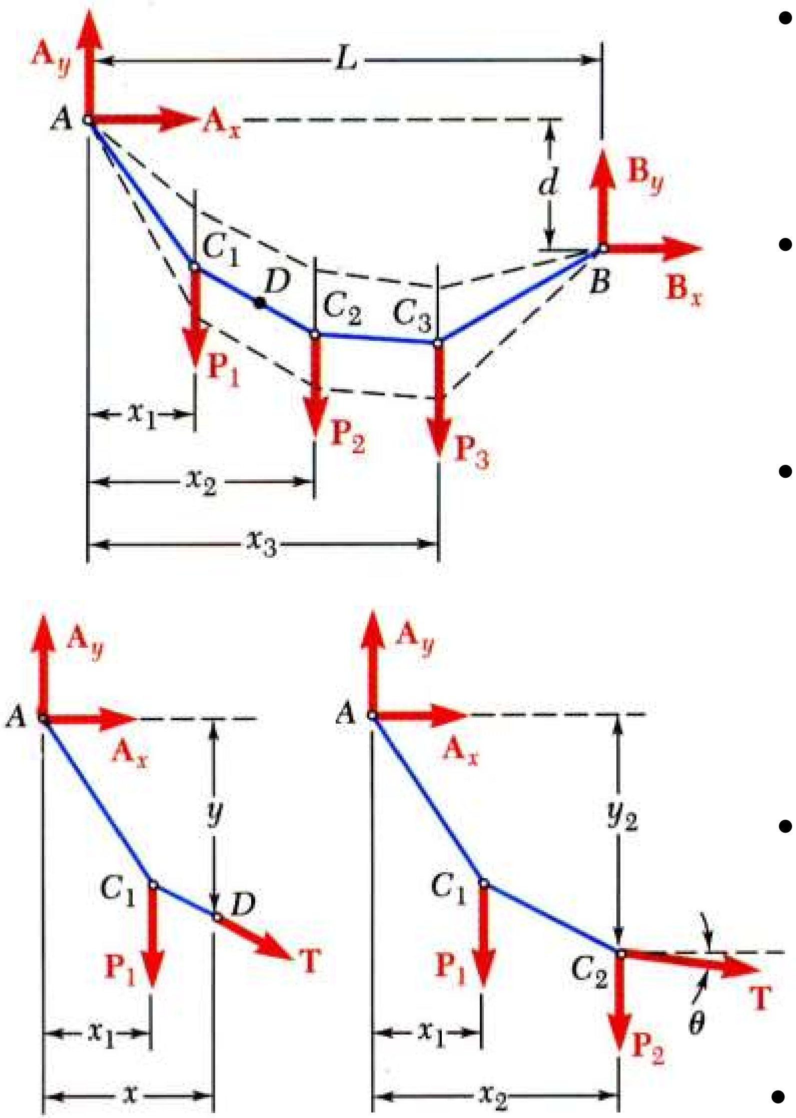
Shear and Bending-Moment Diagrams. The shear at any section is constant and equal to T/L. Since a couple is applied at B, the bending-moment diagram is discontinuous at B; the bending moment decreases suddenly by an amount equal to T.

Cables With Concentrated Loads



- Cables are applied as structural elements in suspension bridges, transmission lines, aerial tramways, guy wires for high towers, etc.
- For analysis, assume:
  a) concentrated vertical loads on given vertical lines,
  b) weight of cable is negligible,
  c) cable is flexible, i.e., resistance to bending is small,
  d) portions of cable between successive loads may be treated as two force members
- Wish to determine shape of cable, i.e., vertical distance from support *A* to each load point.

Cables With Concentrated Loads

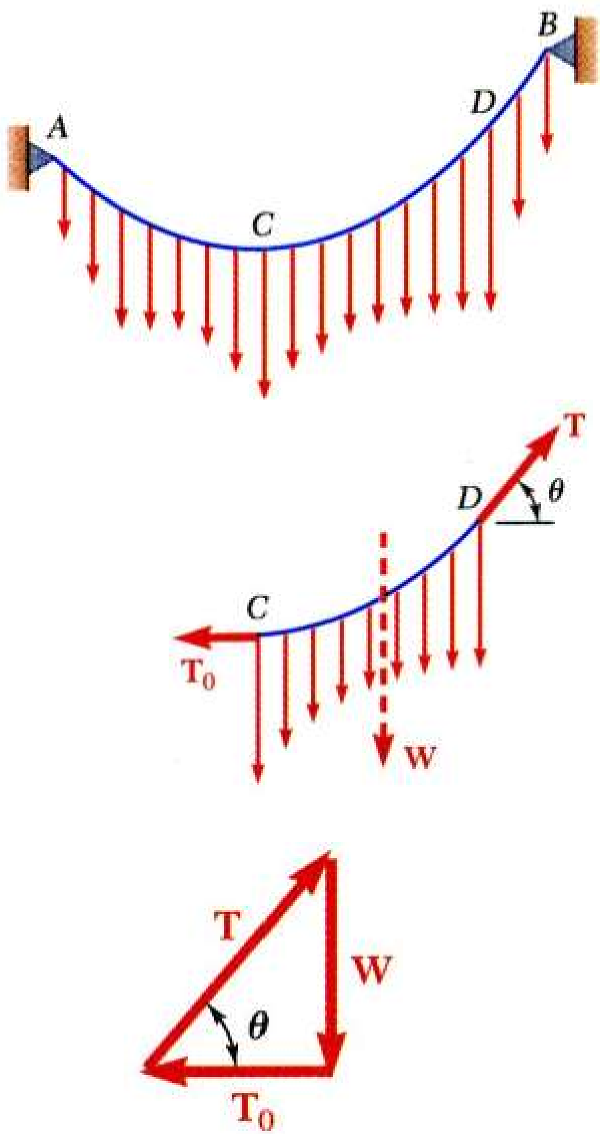


- Consider entire cable as free-body. Slopes of cable at *A* and *B* are not known - two reaction components required at each support.
- Four unknowns are involved and three equations of equilibrium are not sufficient to determine the reactions.
- Additional equation is obtained by considering equilibrium of portion of cable *AD* and assuming that coordinates of point *D* on the cable are known. The additional equation is $\sum M_D = 0$.
- For other points on cable,

  $\sum M_{C_2} = 0$ yields y_2

  $\sum F_x = 0, \sum F_y = 0$ yield T_x, T_y
- $T_x = T\cos\theta = A_x = \text{constant}$

Cables With Distributed Loads



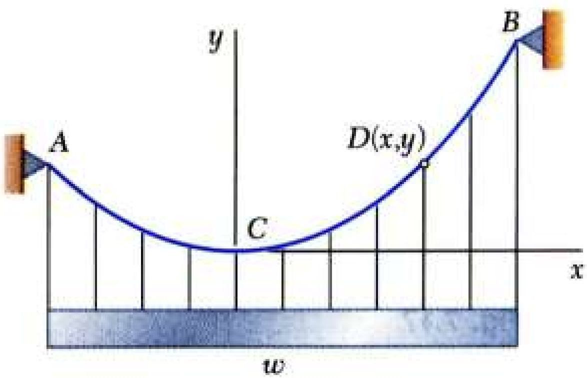
- For cable carrying a distributed load:
  a) cable hangs in shape of a curve
  b) internal force is a tension force directed along tangent to curve.
- Consider free-body for portion of cable extending from lowest point *C* to given point *D*. Forces are horizontal force $\boldsymbol{T_0}$ at C and tangential force $\boldsymbol{T}$ at *D*.
- From force triangle:

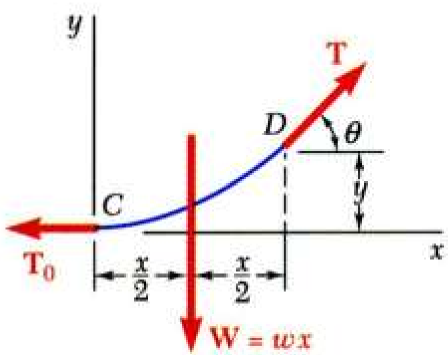
$$T\cos\theta = T_0 \qquad T\sin\theta = W$$

$$T = \sqrt{T_0^2 + W^2} \qquad \tan\theta = \frac{W}{T_0}$$

- Horizontal component of $\boldsymbol{T}$ is uniform over cable.
- Vertical component of $\boldsymbol{T}$ is equal to magnitude of W measured from lowest point.
- Tension is minimum at lowest point and maximum at *A* and *B*.

Parabolic Cable





- Consider a cable supporting a uniform, horizontally distributed load, e.g., support cables for a suspension bridge.

- With loading on cable from lowest point *C* to a point *D* given by $W = wx$, internal tension force magnitude and direction are

$$T = \sqrt{T_0^2 + w^2 x^2} \qquad \tan\theta = \frac{wx}{T_0}$$

- Summing moments about *D*,

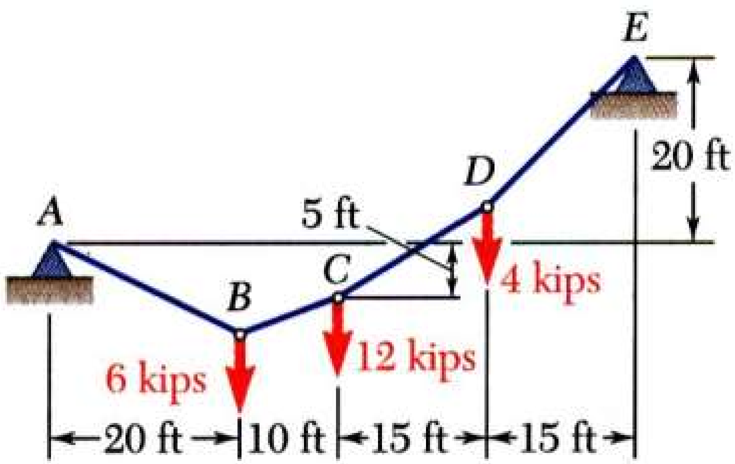
$$\sum M_D = 0: \qquad wx\frac{x}{2} - T_0 y = 0$$

or

$$y = \frac{wx^2}{2T_0}$$

The cable forms a parabolic curve.

Sample Problem 7.8



The cable *AE* supports three vertical loads from the points indicated. If point *C* is 5 ft below the left support, determine *(a)* the elevation of points *B* and *D*, and *(b)* the maximum slope and maximum tension in the cable.

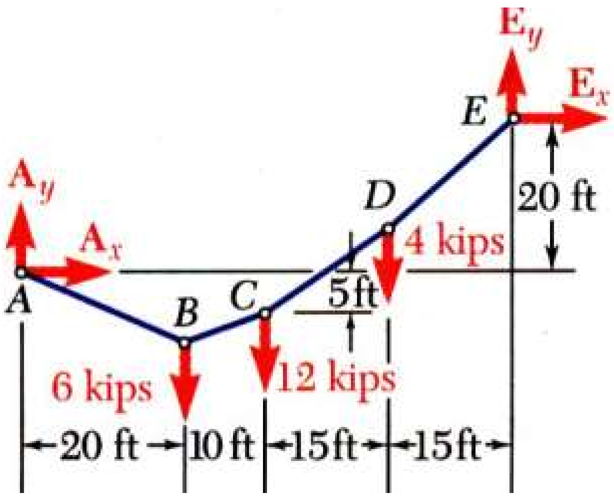
SOLUTION:

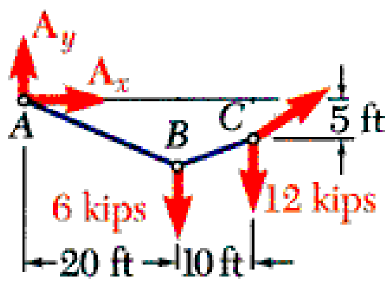
- Determine reaction force components at *A* from solution of two equations formed from taking entire cable as free-body and summing moments about *E*, and from taking cable portion *ABC* as a free-body and summing moments about *C*.

- Calculate elevation of *B* by considering *AB* as a free-body and summing moments *B*. Similarly, calculate elevation of *D* using *ABCD* as a free-body.

- Evaluate maximum slope and maximum tension which occur in *DE*.

Sample Problem 7.8





SOLUTION:

- Determine two reaction force components at *A* from solution of two equations formed from taking entire cable as a free-body and summing moments about *E*,

$$\sum M_E = 0:$$

$$20A_x - 60A_y + 40(6) + 30(12) + 15(4) = 0$$

$$20A_x - 60A_y + 660 = 0$$

and from taking cable portion *ABC* as a free-body and summing moments about *C*.

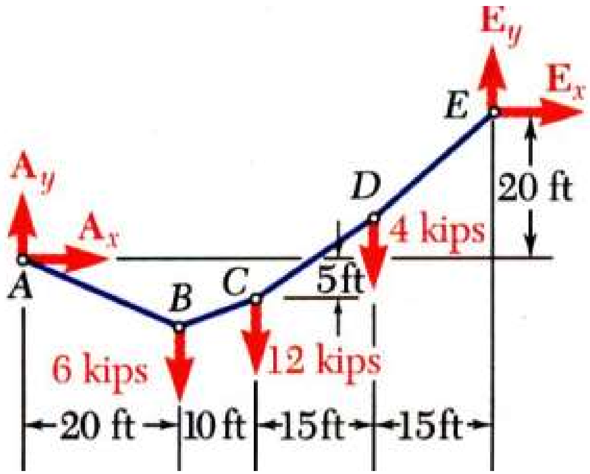
$$\sum M_C = 0:$$

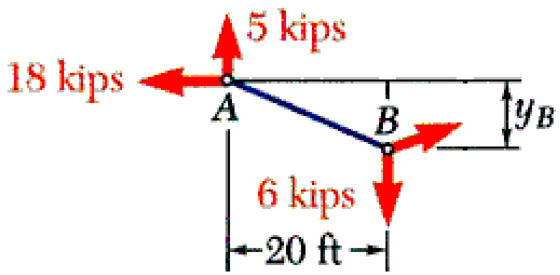
$$-5A_x - 30A_y + 10(6) = 0$$

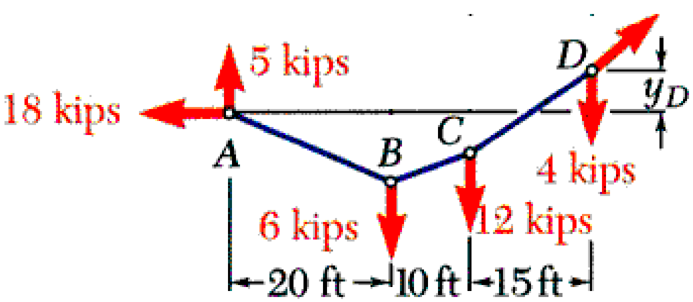
Solving simultaneously,

$$A_x = -18\,\text{kips} \qquad A_y = 5\,\text{kips}$$

Sample Problem 7.8







- Calculate elevation of *B* by considering *AB* as a free-body and summing moments *B*.

$$\sum M_B = 0: \qquad y_B(18) - 5(20) = 0$$

$$\boxed{y_B = -5.56 \text{ ft}}$$

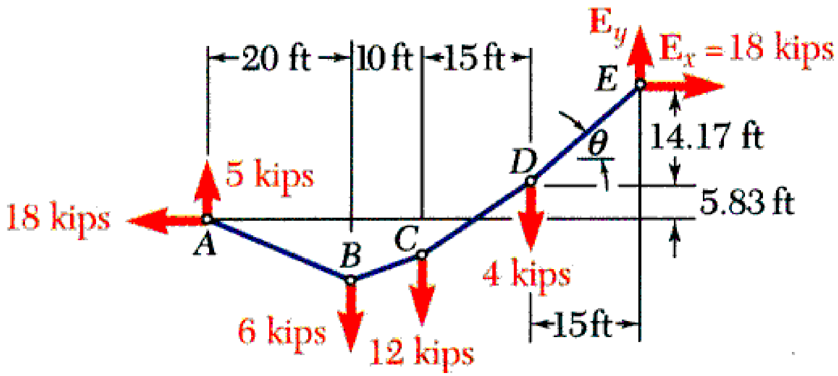
Similarly, calculate elevation of *D* using *ABCD* as a free-body.

$$\sum M = 0:$$

$$-y_D(18) - 45(5) + 25(6) + 15(12) = 0$$

$$\boxed{y_D = 5.83 \text{ ft}}$$

Sample Problem 7.8



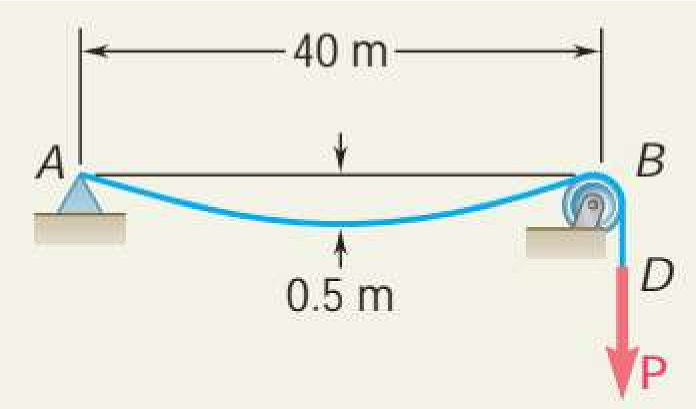
- Evaluate maximum slope and maximum tension which occur in *DE*.

$$\tan\theta = \frac{14.7}{15} \qquad \boxed{\theta = 43.4^\circ}$$

$$T_{\max} = \frac{18\,\text{kips}}{\cos\theta} \qquad \boxed{T_{\max} = 24.8\,\text{kips}}$$

Sample Problem 7.9

A light cable is attached to a support at *A*, passes over a small pulley at *B*, and supports a load **P**. Knowing that the sag of the cable is 0.5 m and that the mass per unit length of the cable is 0.75 kg/m, determine (*a*) the magnitude of the load **P**, (*b*) the slope of the cable at *B*, (*c*) the total length of the cable from *A* to *B*. Since the ratio of the sag to the span is small, assume the cable to be parabolic. Also, neglect the weight of the portion of cable from *B* to *D*.



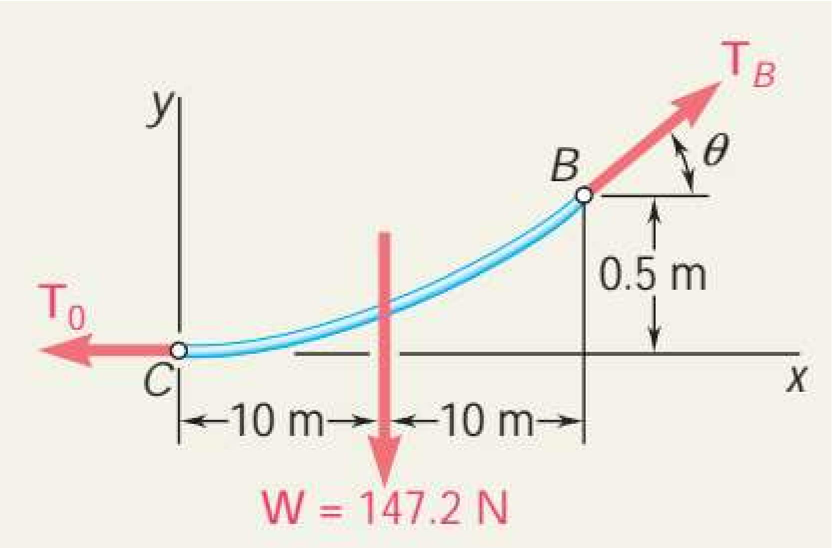
Sample Problem 7.9

a. Load P. We denote by C the lowest point of the cable and draw the free-body diagram of the portion CB of cable. Assuming the load to be uniformly distributed along the horizontal, we write

$$w = (0.75 \text{ kg/m})(9.81 \text{ m/s}^2) = 7.36 \text{ N/m}$$

The total load for the portion CB of cable is

$$W = wx_B = (7.36 \text{ N/m})(20 \text{ m}) = 147.2 \text{ N}$$



Sample Problem 7.9

and is applied halfway between C and B. Summing moments about B, we write

$$+\circlearrowleft \Sigma M_B = 0: \qquad (147.2\text{ N})(10\text{ m}) - T_0(0.5\text{ m}) = 0 \qquad T_0 = 2944\text{ N}$$

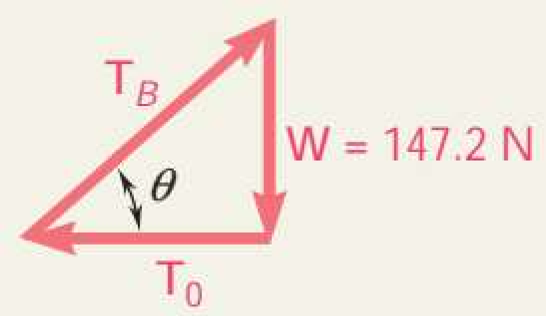
From the force triangle we obtain

$$T_B = \sqrt{T_0^2 + W^2}$$
$$= \sqrt{(2944\text{ N})^2 + (147.2\text{ N})^2} = 2948\text{ N}$$

Since the tension on each side of the pulley is the same, we find

$$P = T_B = 2948\text{ N}$$

Sample Problem 7.9

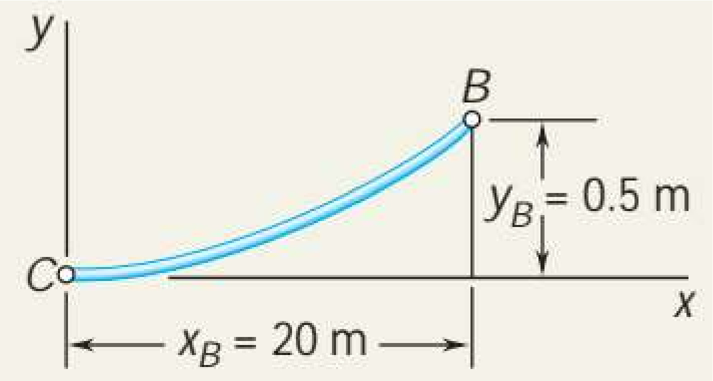


b. Slope of Cable at *B*. We also obtain from the force triangle

$$\tan u = \frac{W}{T_0} = \frac{147.2\ \text{N}}{2944\ \text{N}} = 0.05$$

$$u = 2.9°$$

Sample Problem 7.9



c. Length of Cable. Applying Eq. (7.10) between C and B, we write

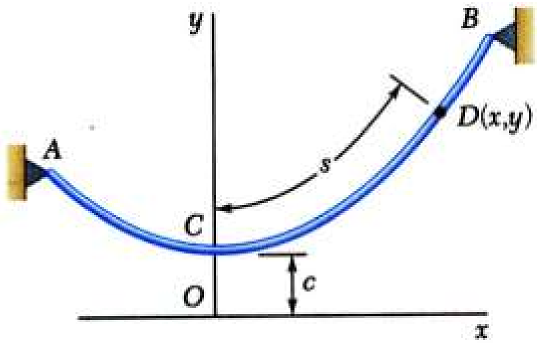
$$s_B = x_B\left[1 + \frac{2}{3}\left(\frac{y_B}{x_B}\right)^2 + \cdots\right]$$

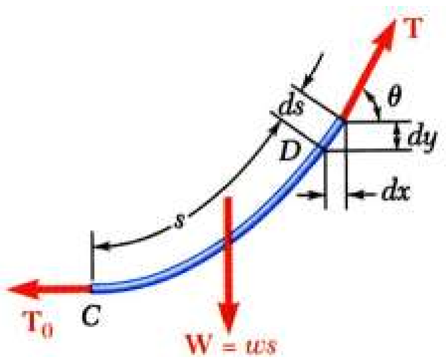
$$= (20 \text{ m})\left[1 + \frac{2}{3}\left(\frac{0.5 \text{ m}}{20 \text{ m}}\right)^2 + \cdots\right] = 20.00833 \text{ m}$$

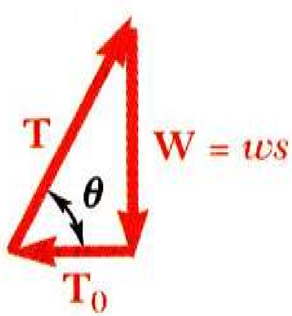
The total length of the cable between A and B is twice this value,

$$\text{Length} = 2s_B = 40.0167 \text{ m}$$

Catenary







- Consider a cable uniformly loaded along the cable itself, e.g., cables hanging under their own weight.
- With loading on the cable from lowest point C to a point D given by $W = ws$, the internal tension force magnitude is

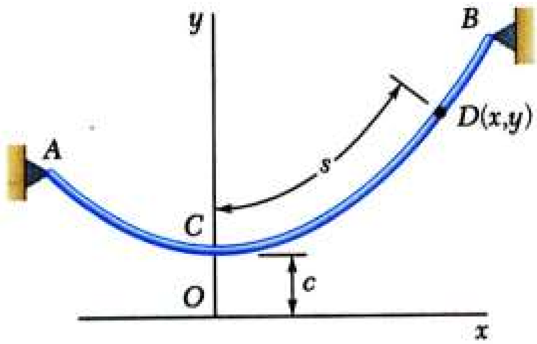
$$T = \sqrt{T_0^2 + w^2 s^2} = w\sqrt{c^2 + s^2} \qquad c = T_0 / w$$

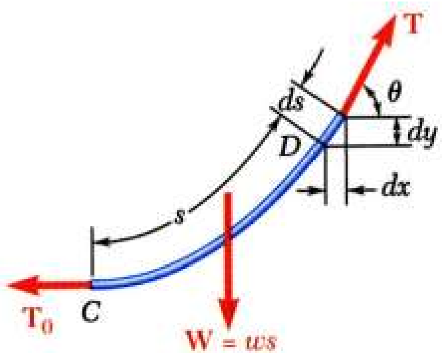
- To relate horizontal distance x to cable length s,

$$dx = ds\cos\theta = \frac{T_0}{T}\cos\theta = \frac{ds}{\sqrt{q + s^2 / c^2}}$$

$$x = \int_0^s \frac{ds}{\sqrt{q + s^2 / c^2}} = c\sinh^{-1}\frac{s}{c} \quad \text{and} \quad s = c\sinh\frac{x}{c}$$

Catenary





- To relate x and y cable coordinates,

$$dy = dx \tan\theta = \frac{W}{T_0} dx = \frac{s}{c} dx = \sinh \frac{x}{c} dx$$

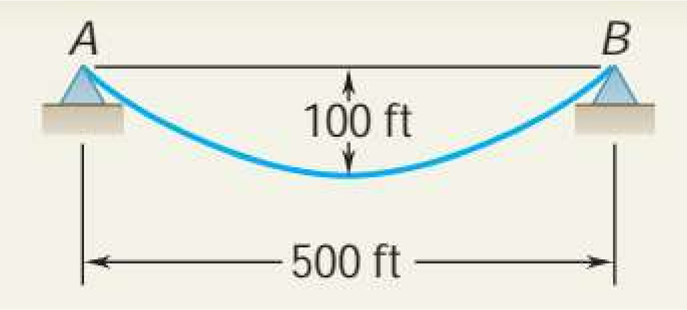
$$y - c = \int_0^x \sinh \frac{x}{c} dx = c \cosh \frac{x}{c} - c$$

$$y = c \cosh \frac{x}{c}$$

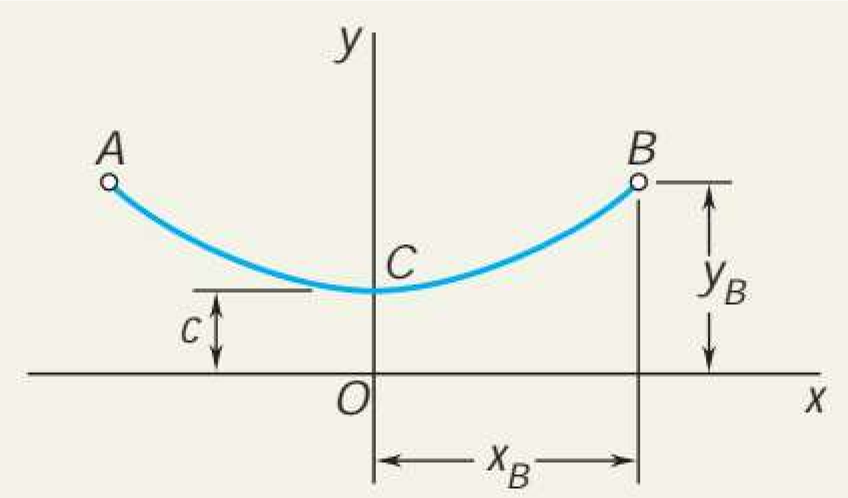
which is the equation of a catenary.

Sample Problem 7.10

A uniform cable weighing 3 lb/ft is suspended between two points *A* and *B* as shown. Determine (*a*) the maximum and minimum values of the tension in the cable, (*b*) the length of the cable.



Sample Problem 7.10



Equation of Cable. The origin of coordinates is placed at a distance c below the lowest point of the cable. The equation of the cable is given by Eq. (7.16),

$$y = c \cosh \frac{x}{c}$$

The coordinates of point B are

$$x_B = 250 \text{ ft} \qquad y_B = 100 + c$$

Sample Problem 7.10

Substituting these coordinates into the equation of the cable, we obtain

$$100 + c = c \cosh \frac{250}{c}$$

$$\frac{100}{c} + 1 = \cosh \frac{250}{c}$$

The value of c is determined by assuming successive trial values, as shown in the following table:

c	$\frac{250}{c}$	$\frac{100}{c}$	$\frac{100}{c} + 1$	$\cosh \frac{250}{c}$
300	0.833	0.333	1.333	1.367
350	0.714	0.286	1.286	1.266
330	0.758	0.303	1.303	1.301
328	0.762	0.305	1.305	1.305

Sample Problem 7.10

Taking $c = 328$, we have

$$y_B = 100 + c = 428 \text{ ft}$$

a. Maximum and Minimum Values of the Tension. Using Eqs. (7.18), we obtain

$$T_{\min} = T_0 = wc = (3 \text{ lb/ft})(328 \text{ ft}) \qquad T_{\min} = 984 \text{ lb}$$
$$T_{\max} = T_B = wy_B = (3 \text{ lb/ft})(428 \text{ ft}) \qquad T_{\max} = 1284 \text{ lb}$$

b. Length of Cable. One-half the length of the cable is found by solving Eq. (7.17):

$$y_B^2 - s_{CB}^2 = c^2 \qquad s_{CB}^2 = y_B^2 - c^2 = (428)^2 - (328)^2 \qquad s_{CB} = 275 \text{ ft}$$

The total length of the cable is therefore

$$s_{AB} = 2s_{CB} = 2(275 \text{ ft}) \qquad s_{AB} = 550 \text{ ft}$$

CHAPTER

8

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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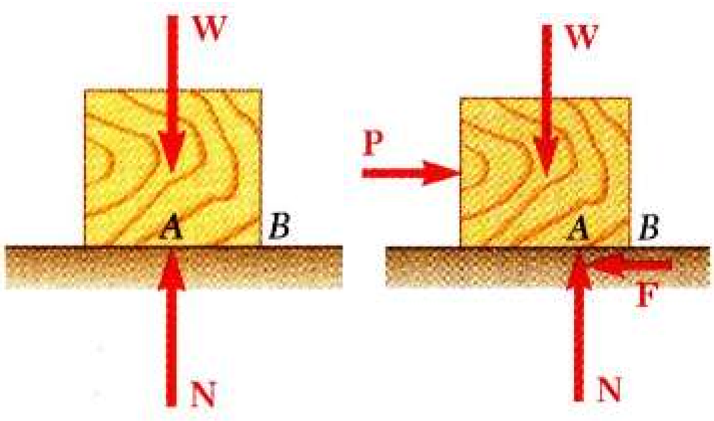
Friction

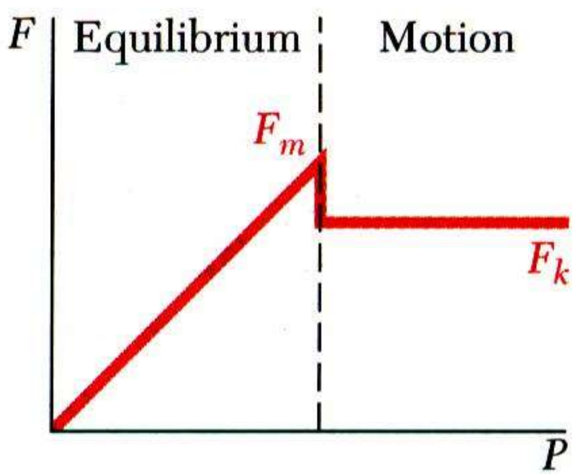
Contents

Introduction

- In preceding chapters, it was assumed that surfaces in contact were either *frictionless* (surfaces could move freely with respect to each other) or *rough* (tangential forces prevent relative motion between surfaces).
- Actually, no perfectly frictionless surface exists. For two surfaces in contact, tangential forces, called *friction forces*, will develop if one attempts to move one relative to the other.
- However, the friction forces are limited in magnitude and will not prevent motion if sufficiently large forces are applied.
- The distinction between frictionless and rough is, therefore, a matter of degree.
- There are two types of friction: *dry* or *Coulomb friction* and *fluid friction*. Fluid friction applies to lubricated mechanisms. The present discussion is limited to dry friction between nonlubricated surfaces.

The Laws of Dry Friction. Coefficients of Friction





- Block of weight *W* placed on horizontal surface. Forces acting on block are its weight and reaction of surface *N*.

- Small horizontal force *P* applied to block. For block to remain stationary, in equilibrium, a horizontal component *F* of the surface reaction is required. *F* is a *static-friction force*.

- As *P* increases, the static-friction force *F* increases as well until it reaches a maximum value F_m.

$$F_m = \mu_s N$$

- Further increase in P causes the block to begin to move as *F* drops to a smaller *kinetic-friction force* F_k.

$$F_k = \mu_k N$$

The Laws of Dry Friction. Coefficients of Friction

Table 8.1. Approximate Values of Coefficient of Static Friction for Dry Surfaces

Metal on metal	0.15–0.60
Metal on wood	0.20–0.60
Metal on stone	0.30–0.70
Metal on leather	0.30–0.60
Wood on wood	0.25–0.50
Wood on leather	0.25–0.50
Stone on stone	0.40–0.70
Earth on earth	0.20–1.00
Rubber on concrete	0.60–0.90

- Maximum static-friction force:

$$F_m = \mu_s N$$

- Kinetic-friction force:

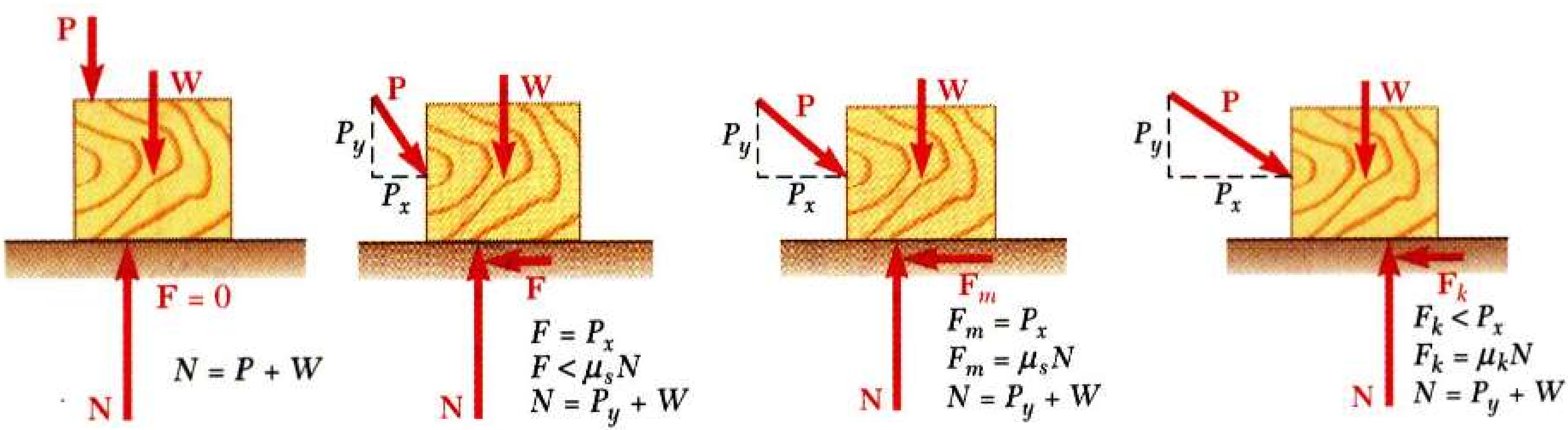
$$F_k = \mu_k N$$

$$\mu_k \cong 0.75\mu_s$$

- Maximum static-friction force and kinetic-friction force are:
  - proportional to normal force
  - dependent on type and condition of contact surfaces
  - independent of contact area

The Laws of Dry Friction. Coefficients of Friction

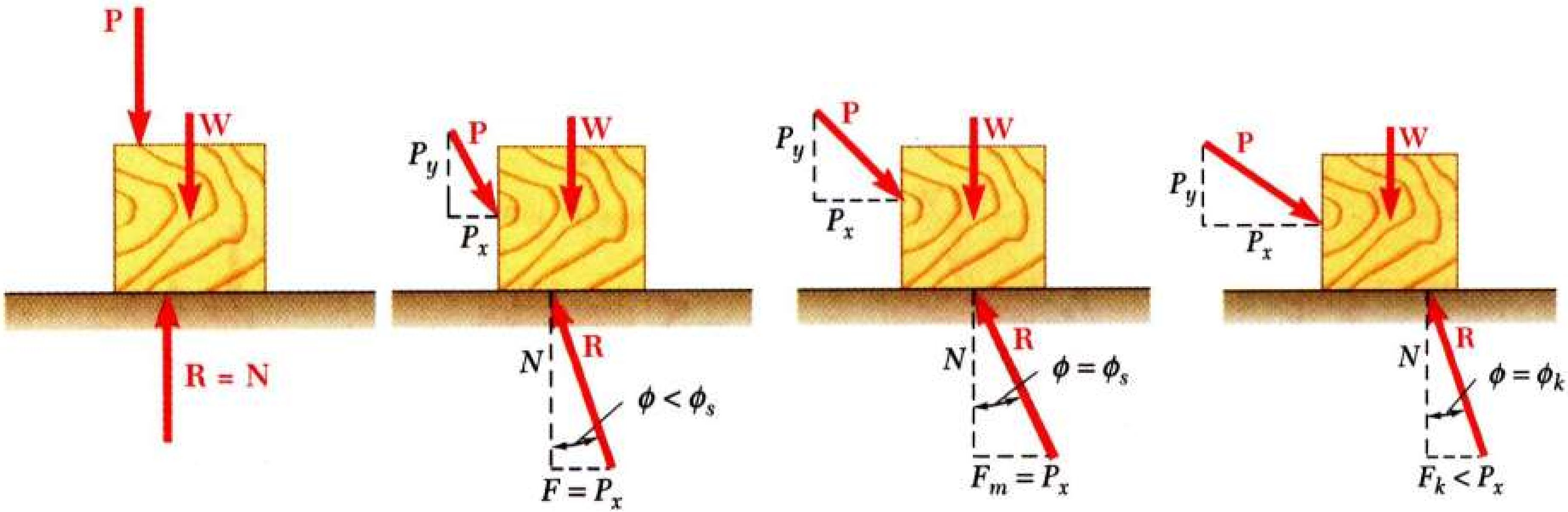
- Four situations can occur when a rigid body is in contact with a horizontal surface:



- No friction, ($P_x = 0$)
- No motion, ($P_x < F_m$)
- Motion impending, ($P_x = F_m$)
- Motion, ($P_x > F_m$)

Angles of Friction

- It is sometimes convenient to replace normal force N and friction force F by their resultant $\boldsymbol{R}$:



- No friction
- No motion
- Motion impending
- Motion

$$\tan\phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N}$$

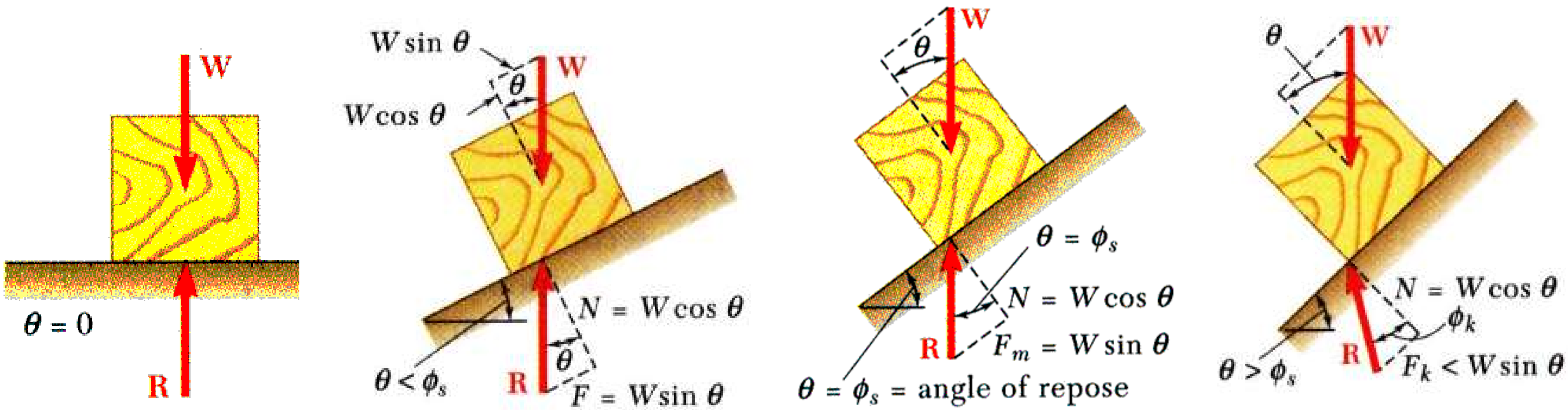
$$\tan\phi_s = \mu_s$$

$$\tan\phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N}$$

$$\tan\phi_k = \mu_k$$

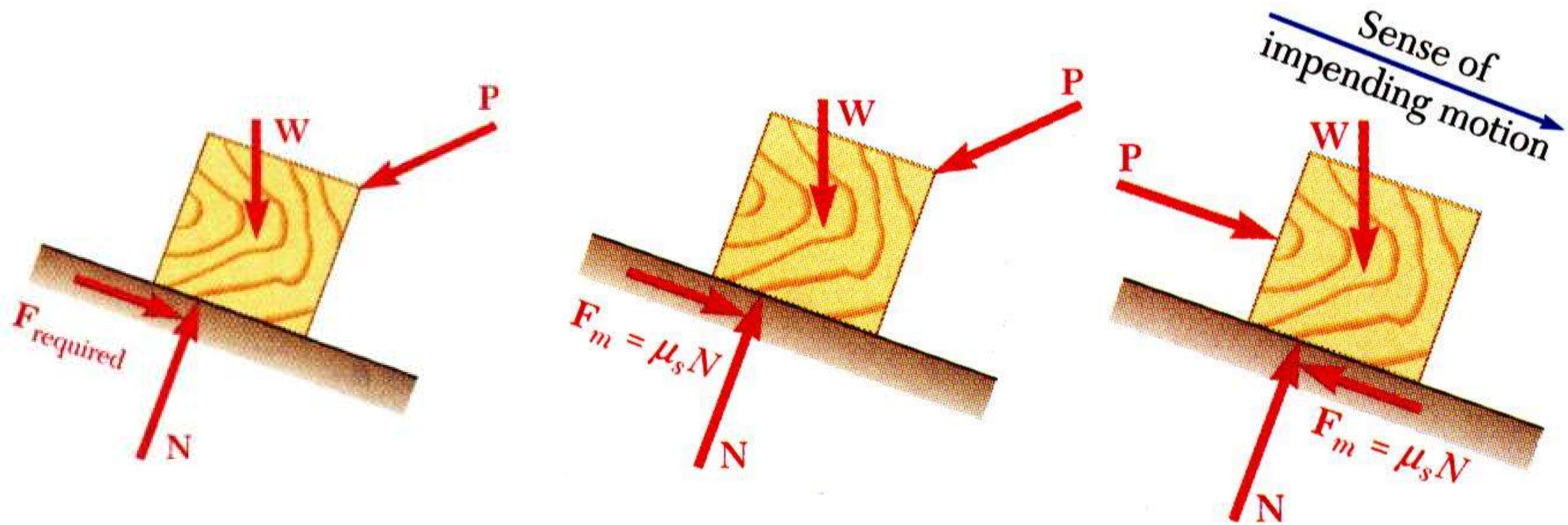
Angles of Friction

- Consider block of weight *W* resting on board with variable inclination angle *θ*.



- No friction
- No motion
- Motion impending
- Motion

Problems Involving Dry Friction

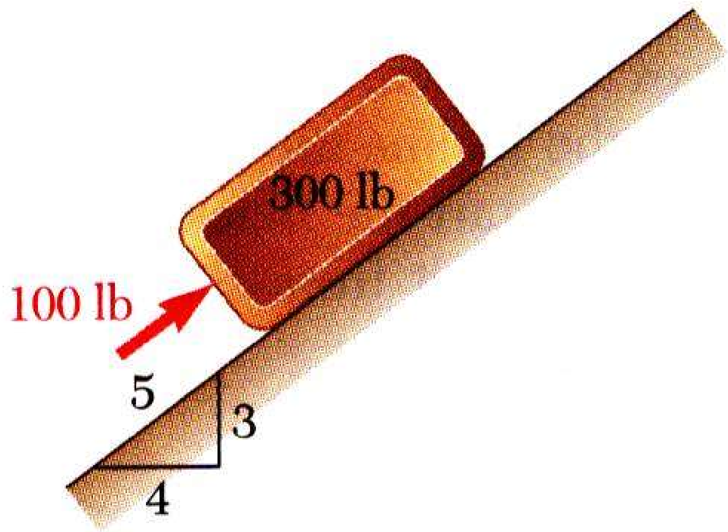


- All applied forces known
- Coefficient of static friction is known
- Determine whether body will remain at rest or slide

- All applied forces known
- Motion is impending
- Determine value of coefficient of static friction.

- Coefficient of static friction is known
- Motion is impending
- Determine magnitude or direction of one of the applied forces

Sample Problem 8.1

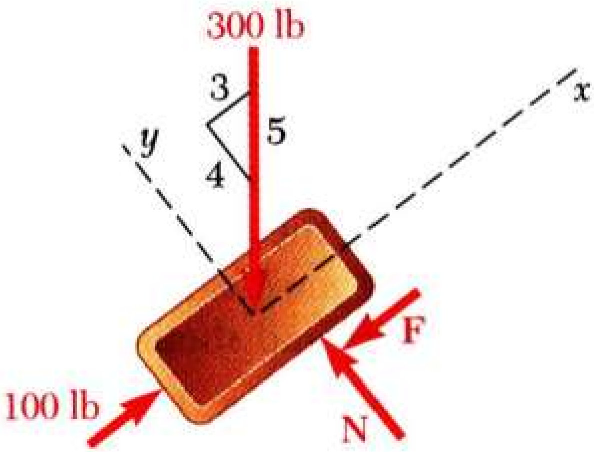


A 100 lb force acts as shown on a 300 lb block placed on an inclined plane. The coefficients of friction between the block and plane are $\mu_s = 0.25$ and $\mu_k = 0.20$. Determine whether the block is in equilibrium and find the value of the friction force.

SOLUTION:

- Determine values of friction force and normal reaction force from plane required to maintain equilibrium.
- Calculate maximum friction force and compare with friction force required for equilibrium. If it is greater, block will not slide.
- If maximum friction force is less than friction force required for equilibrium, block will slide. Calculate kinetic-friction force.

Sample Problem 8.1



SOLUTION:

- Determine values of friction force and normal reaction force from plane required to maintain equilibrium.

$$\sum F_x = 0: \qquad 100\,\text{lb} - \tfrac{3}{5}(300\,\text{lb}) - F = 0$$

$$F = -80\,\text{lb}$$

$$\sum F_y = 0: \qquad N - \tfrac{4}{5}(300\,\text{lb}) = 0$$

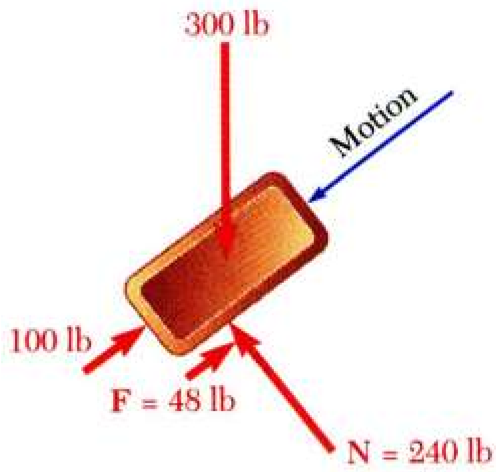
$$N = 240\,\text{lb}$$

- Calculate maximum friction force and compare with friction force required for equilibrium. If it is greater, block will not slide.

$$F_m = \mu_s N \qquad F_m = 0.25(240\,\text{lb}) = 48\,\text{lb}$$

The block will slide down the plane.

Sample Problem 8.1



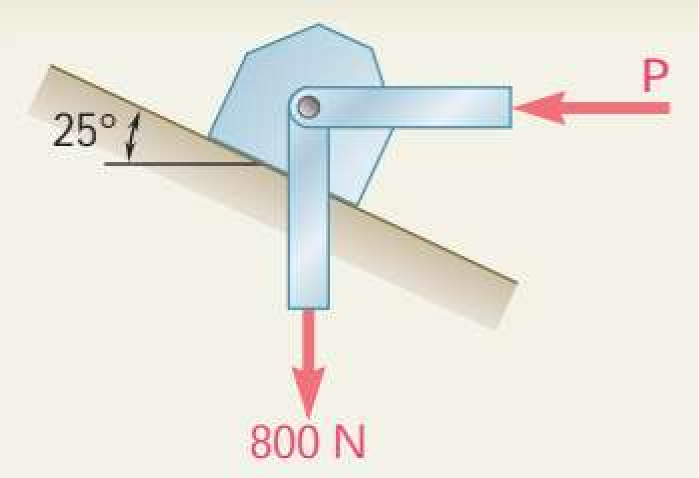
- If maximum friction force is less than friction force required for equilibrium, block will slide. Calculate kinetic-friction force.

$$F_{actual} = F_k = \mu_k N$$
$$= 0.20(240\,\text{lb})$$

$$\boxed{F_{actual} = 48\,\text{lb}}$$

Sample Problem 8.2

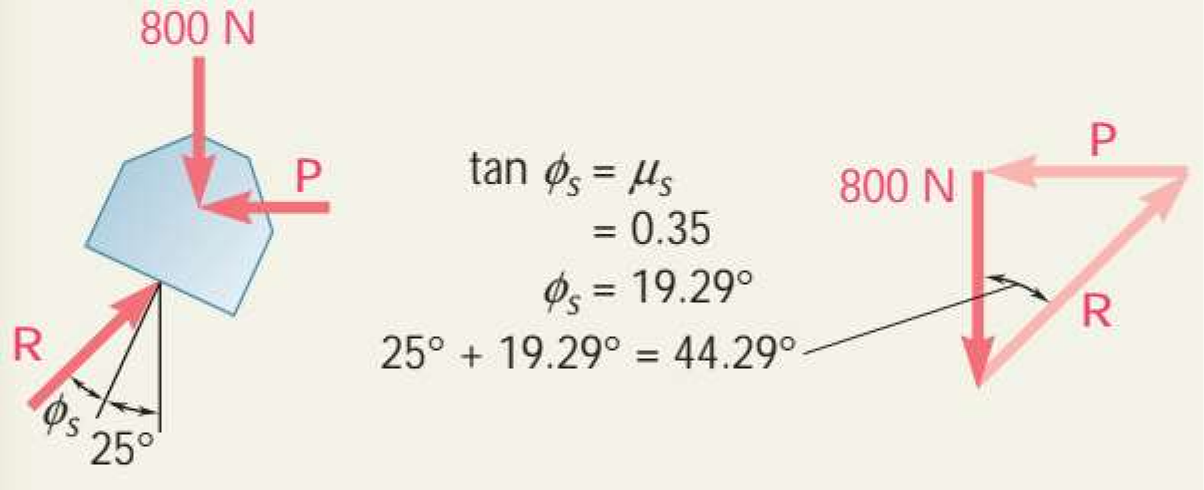
A support block is acted upon by two forces as shown. Knowing that the coefficients of friction between the block and the incline are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the force **P** required (*a*) to start the block moving up the incline, (*b*) to keep it moving up, (*c*) to prevent it from sliding down.



Sample Problem 8.2

Free-Body Diagram. For each part of the problem we draw a free-body diagram of the block and a force triangle including the 800-N vertical force, the horizontal force **P**, and the force **R** exerted on the block by the incline. The direction of **R** must be determined in each separate case. We note that since **P** is perpendicular to the 800-N force, the force triangle is a right triangle, which can easily be solved for **P**. In most other problems, however, the force triangle will be an oblique triangle and should be solved by applying the law of sines.

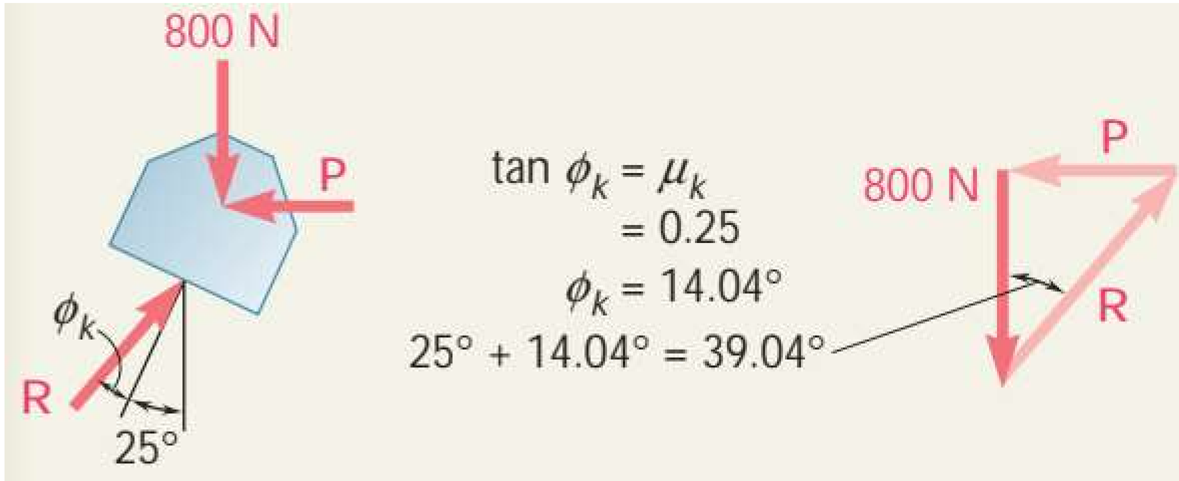
Sample Problem 8.2



a. Force *P* to Start Block Moving Up

$$P = (800\ \text{N}) \tan 44.29° \qquad \mathbf{P} = 780\ \text{N} z$$

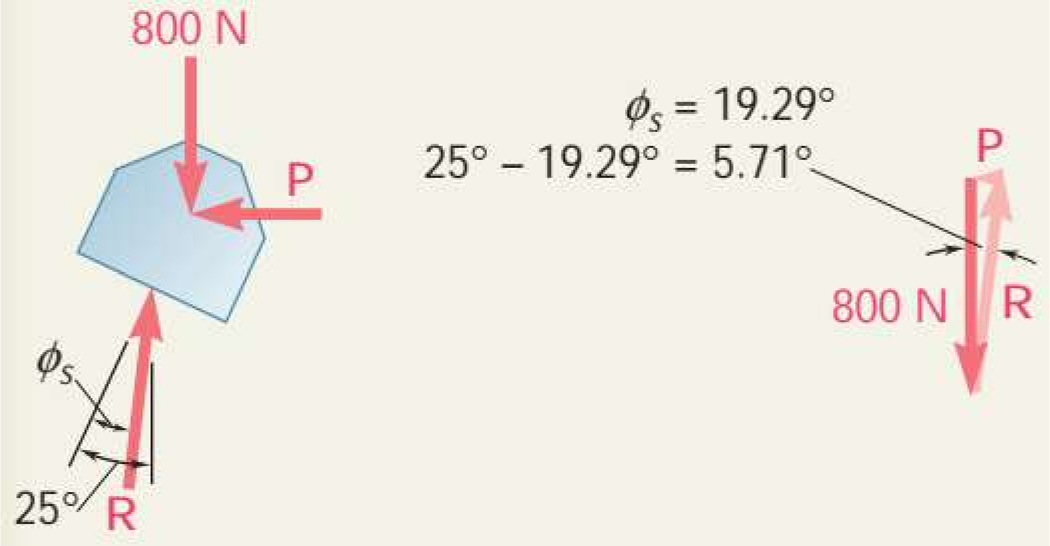
Sample Problem 8.2



b. Force *P* to Keep Block Moving Up

$$P = (800 \text{ N}) \tan 39.04° \qquad \mathbf{P} = 649 \text{ N}z$$

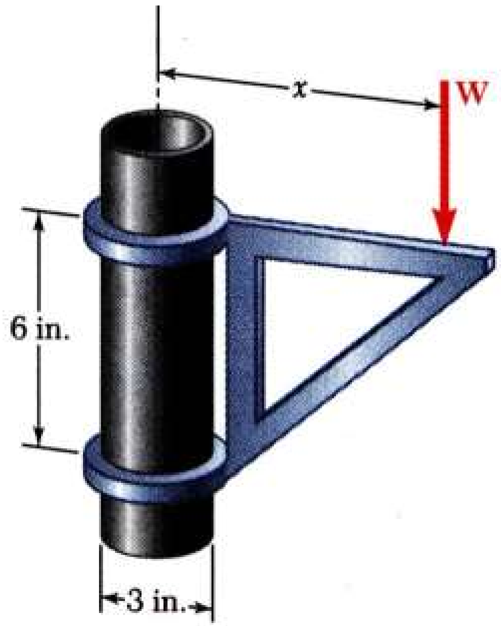
Sample Problem 8.2



c. Force *P* to Prevent Block from Sliding Down

$$P = (800 \text{ N}) \tan 5.71° \qquad \mathbf{P} = 80.0 \text{ N} z$$

Sample Problem 8.3

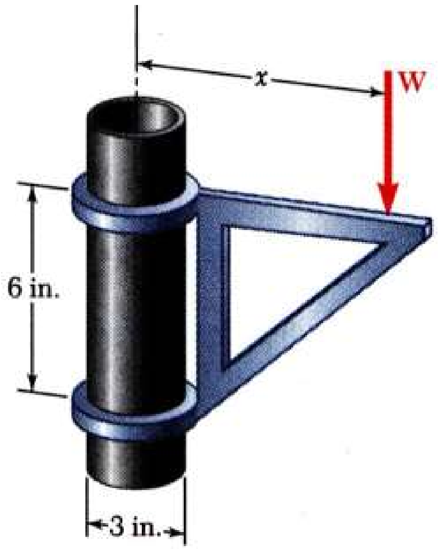


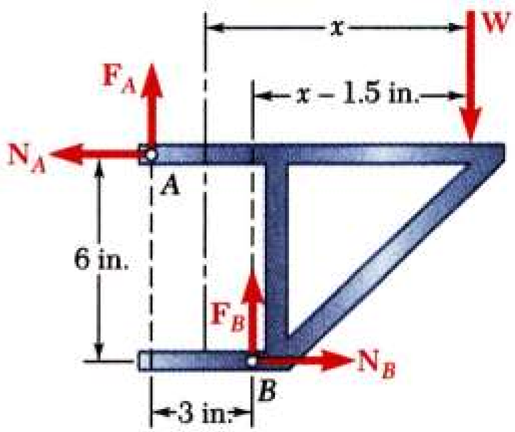
The moveable bracket shown may be placed at any height on the 3-in. diameter pipe. If the coefficient of friction between the pipe and bracket is 0.25, determine the minimum distance *x* at which the load can be supported. Neglect the weight of the bracket.

SOLUTION:

- When *W* is placed at minimum *x*, the bracket is about to slip and friction forces in upper and lower collars are at maximum value.
- Apply conditions for static equilibrium to find minimum *x*.

Sample Problem 8.3





SOLUTION:

- When W is placed at minimum x, the bracket is about to slip and friction forces in upper and lower collars are at maximum value.

$$F_A = \mu_s N_A = 0.25 N_A$$

$$F_B = \mu_s N_B = 0.25 N_B$$

- Apply conditions for static equilibrium to find minimum x.

$$\sum F_x = 0: \quad N_B - N_A = 0 \qquad N_B = N_A$$

$$\sum F_y = 0: \quad F_A + F_B - W = 0$$

$$0.25 N_A + 0.25 N_B - W = 0$$

$$0.5 N_A = W \qquad N_A = N_B = 2W$$

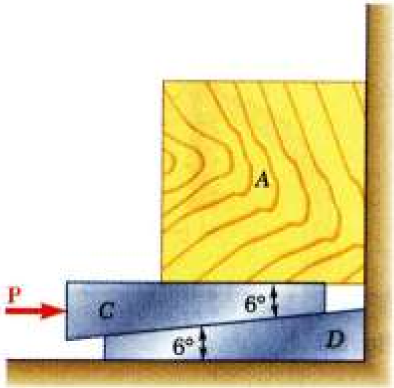
$$\sum M_B = 0: \quad N_A(6\text{ in.}) - F_A(3\text{ in.}) - W(x - 1.5\text{ in.}) = 0$$

$$6N_A - 3(0.25 N_A) - W(x - 1.5) = 0$$

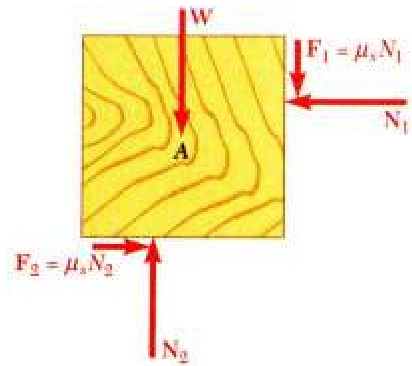
$$6(2W) - 0.75(2W) - W(x - 1.5) = 0$$

$$\boxed{x = 12\text{ in.}}$$

Wedges



- *Wedges* - simple machines used to raise heavy loads.
- Force required to lift block is significantly less than block weight.
- Friction prevents wedge from sliding out.
- Want to find minimum force P to raise block.

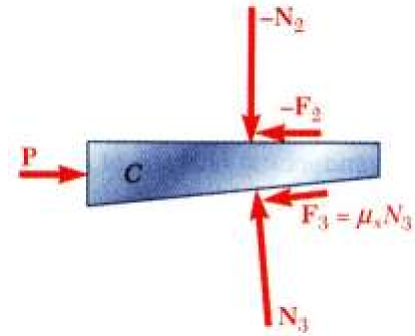


- Block as free-body

$$\sum F_x = 0:$$
$$-N_1 + \mu_s N_2 = 0$$
$$\sum F_y = 0:$$
$$-W - \mu_s N_1 + N_2 = 0$$

or

$$\vec{R}_1 + \vec{R}_2 + \vec{W} = 0$$



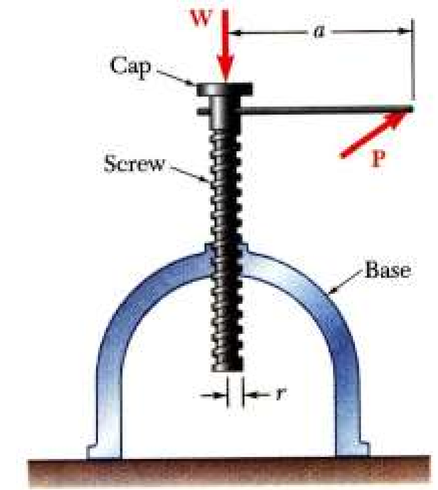
- Wedge as free-body

$$\sum F_x = 0:$$
$$-\mu_s N_2 - N_3(\mu_s \cos 6° - \sin 6°) + P = 0$$
$$\sum F_y = 0:$$
$$-N_2 + N_3(\cos 6° - \mu_s \sin 6°) = 0$$

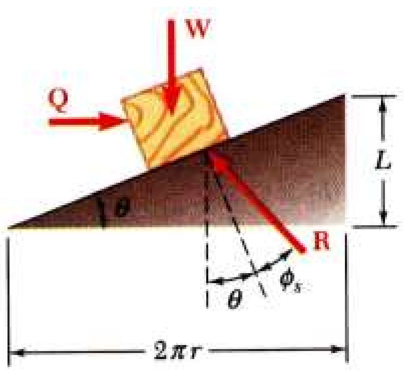
or

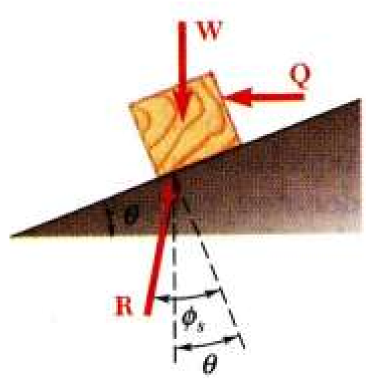
$$\vec{P} - \vec{R}_2 + \vec{R}_3 = 0$$

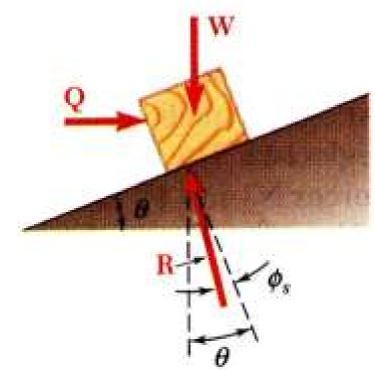
Square-Threaded Screws



- Square-threaded screws frequently used in jacks, presses, etc. Analysis similar to block on inclined plane. Recall friction force does not depend on area of contact.
- Thread of base has been "unwrapped" and shown as straight line. Slope is $2\pi r$ horizontally and lead L vertically.
- Moment of force Q is equal to moment of force P. $Q = Pa/r$



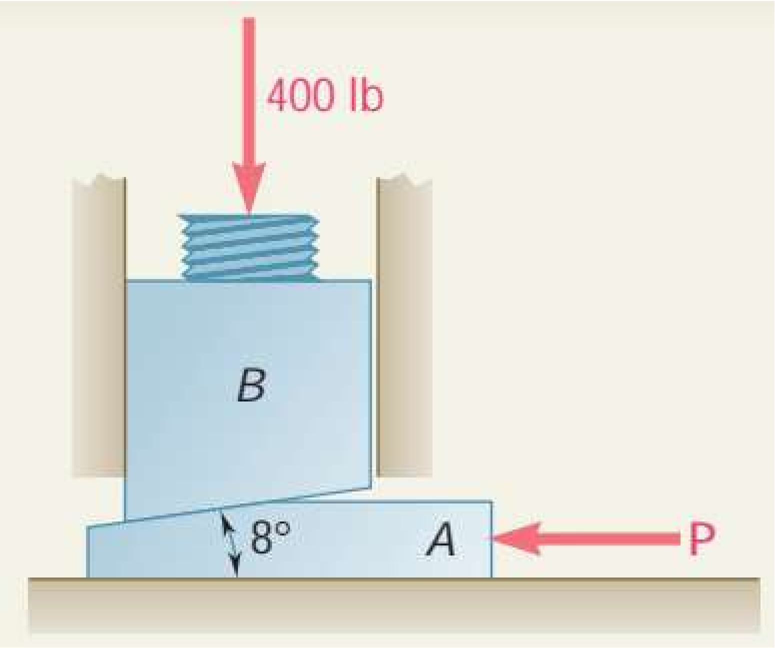




- Impending motion upwards. Solve for Q.
- $\phi_s > \theta$, Self-locking, solve for Q to lower load.
- $\phi_s > \theta$, Non-locking, solve for Q to hold load.

Sample Problem 8.4

The position of the machine block B is adjusted by moving the wedge A. Knowing that the coefficient of static friction is 0.35 between all surfaces of contact, determine the force **P** required (*a*) to raise block B, (*b*) to lower block B.

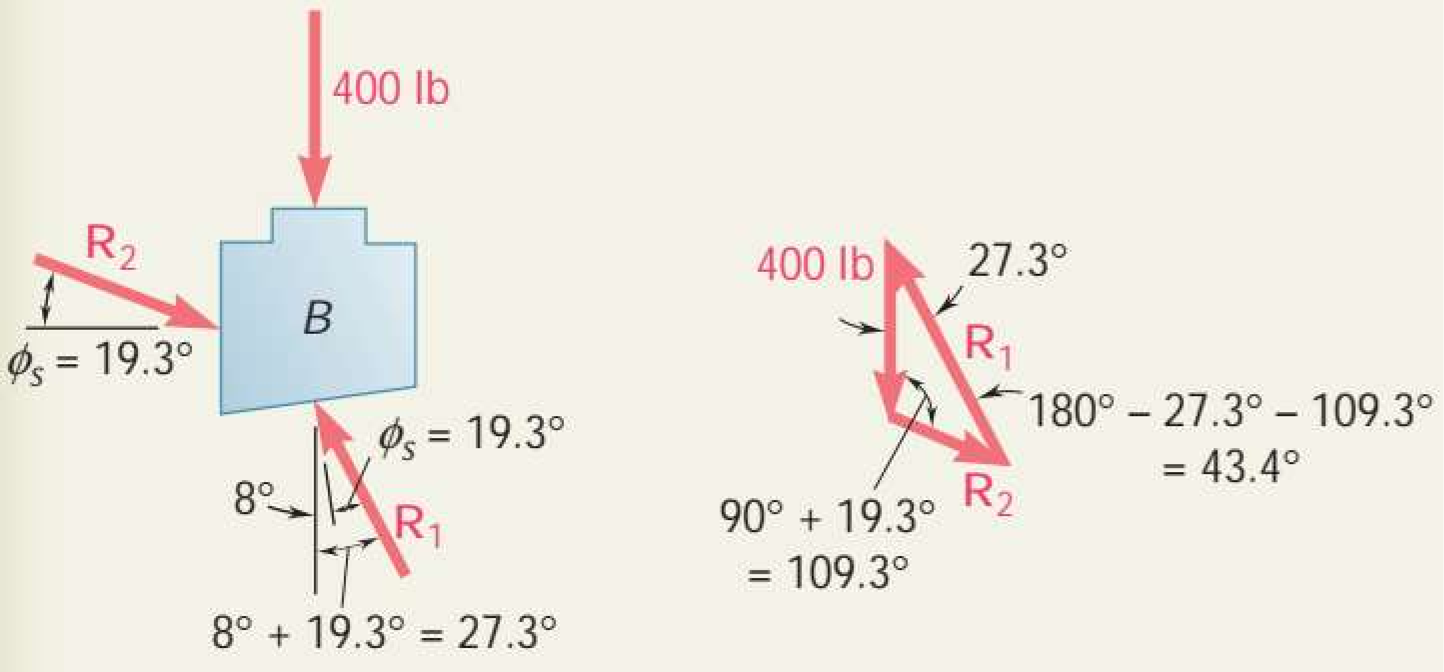


Sample Problem 8.4

For each part, the free-body diagrams of block *B* and wedge *A* are drawn, together with the corresponding force triangles, and the law of sines is used to find the desired forces. We note that since $\mu_s = 0.35$, the angle of friction is

$$\phi_s = \tan^{-1} 0.35 = 19.3°$$

Sample Problem 8.4



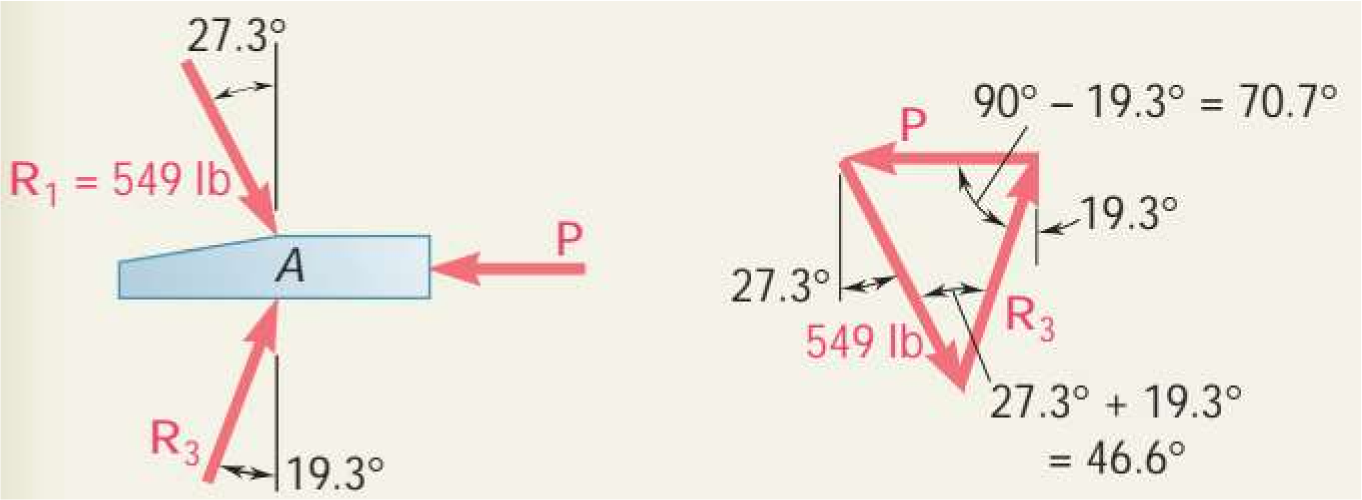
a. Force P to Raise Block

Free Body: Block B

$$\frac{R_1}{\sin 109.3^\circ} = \frac{400\ \text{lb}}{\sin 43.4^\circ}$$

$$R_1 = 549\,\text{lb}$$

Sample Problem 8.4

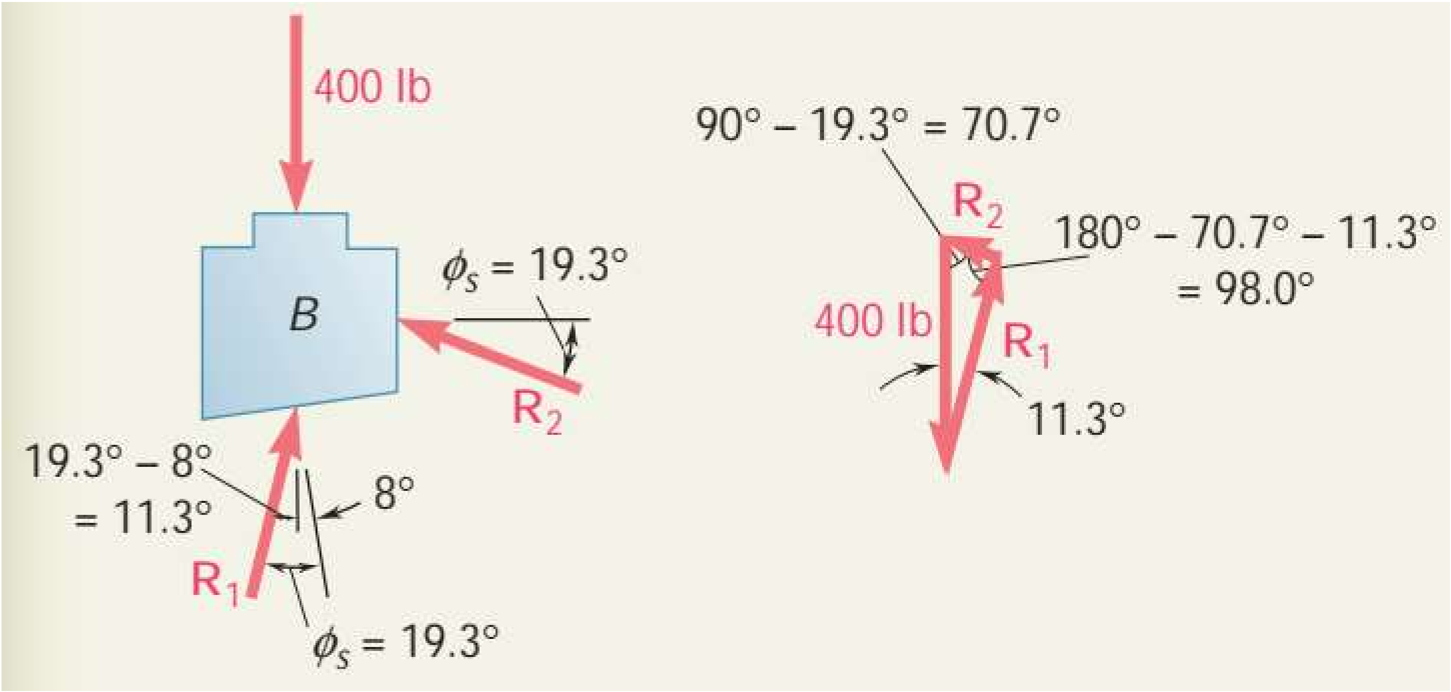


Free Body: Wedge A

$$\frac{P}{\sin 46.6^\circ} = \frac{549 \text{ lb}}{\sin 70.7^\circ}$$

$$P = 423 \text{ lb} \qquad \mathbf{P} = 423 \text{ lb}$$

Sample Problem 8.4



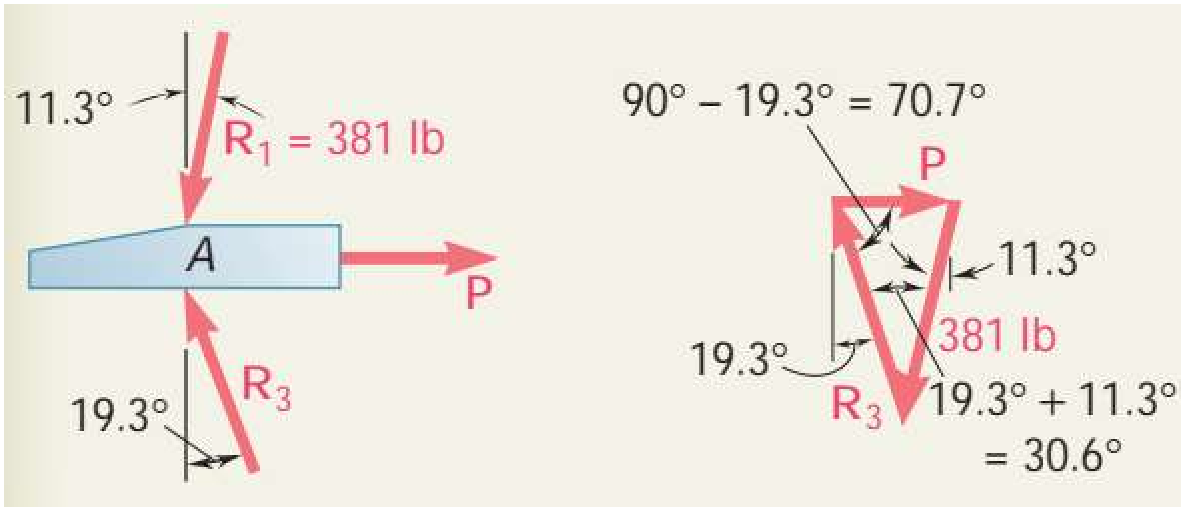
b. Force P to Lower Block

Free Body: Block B

$$\frac{R_1}{\sin 70.7^\circ} = \frac{400 \text{ lb}}{\sin 98.0^\circ}$$

$$R_1 = 381 \text{ lb}$$

Sample Problem 8.4

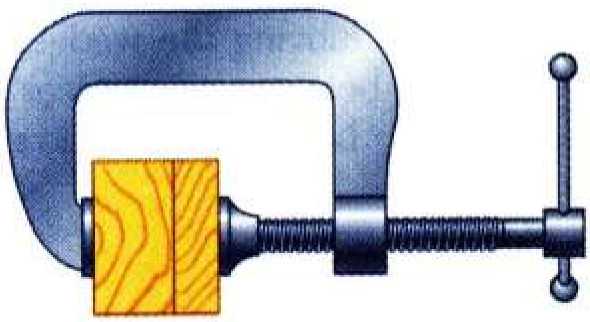


Free Body: Wedge A

$$\frac{P}{\sin 30.6°} = \frac{381 \text{ lb}}{\sin 70.7°}$$

$$P = 206 \text{ lb} \qquad \mathbf{P} = 206 \text{ lb y}$$

Sample Problem 8.5



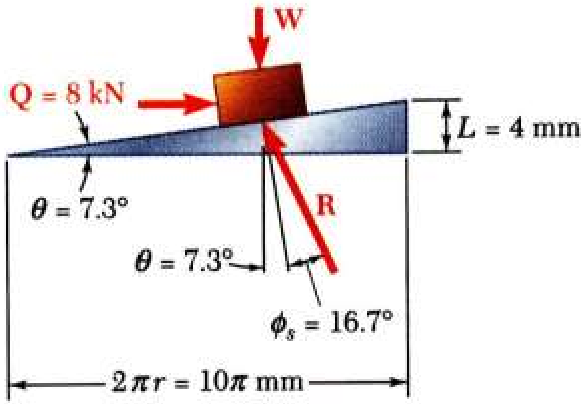
A clamp is used to hold two pieces of wood together as shown. The clamp has a double square thread of mean diameter equal to 10 mm with a pitch of 2 mm. The coefficient of friction between threads is $\mu_s = 0.30$.

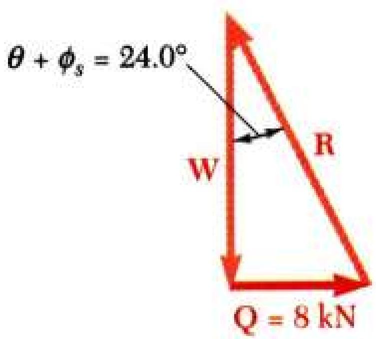
If a maximum torque of 40 N*m is applied in tightening the clamp, determine (a) the force exerted on the pieces of wood, and (b) the torque required to loosen the clamp.

SOLUTION

- Calculate lead angle and pitch angle.
- Using block and plane analogy with impending motion up the plane, calculate the clamping force with a force triangle.
- With impending motion down the plane, calculate the force and torque required to loosen the clamp.

Sample Problem 8.5





SOLUTION

- Calculate lead angle and pitch angle. For the double threaded screw, the lead L is equal to twice the pitch.

$$\tan\theta = \frac{L}{2\pi r} = \frac{2(2\text{ mm})}{10\pi\text{ mm}} = 0.1273 \qquad \theta = 7.3°$$

$$\tan\phi_s = \mu_s = 0.30 \qquad \phi_s = 16.7°$$

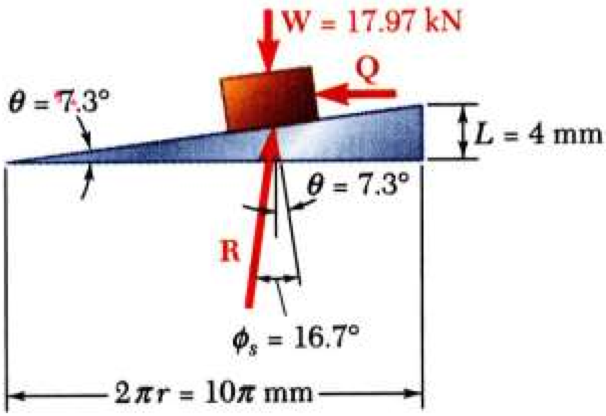
- Using block and plane analogy with impending motion up the plane, calculate clamping force with force triangle.

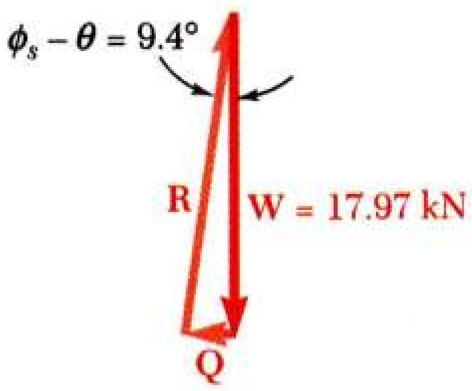
$$Q\,r = 40\text{ N}\cdot\text{m} \qquad Q = \frac{40\text{ N}\cdot\text{m}}{5\text{ mm}} = 8\text{ kN}$$

$$\tan(\theta+\phi_s) = \frac{Q}{W} \qquad W = \frac{8\text{ kN}}{\tan 24°}$$

$$\boxed{W = 17.97\text{ kN}}$$

Sample Problem 8.5





- With impending motion down the plane, calculate the force and torque required to loosen the clamp.

$$\tan(\phi_s - \theta) = \frac{Q}{W} \qquad Q = (17.97\,\text{kN})\tan 9.4^\circ$$

$$Q = 2.975\,\text{kN}$$

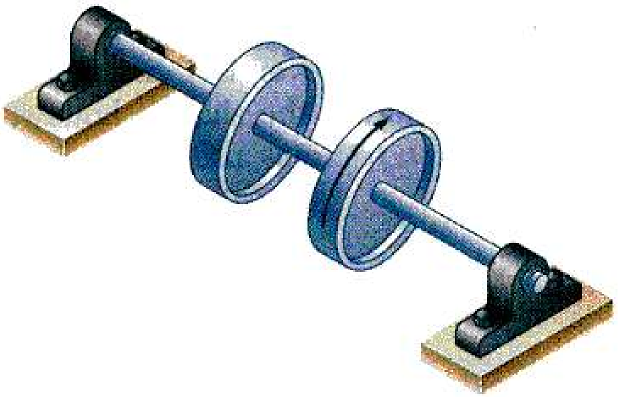
$$Torque = Q\,r = (2.975\,\text{kN})(5\,\text{mm})$$
$$= \left(2.975\times10^{3}\,\text{N}\right)\left(5\times10^{-3}\,\text{m}\right)$$

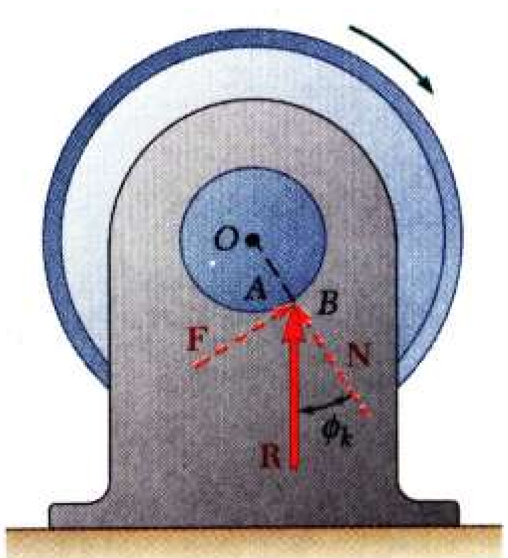
$$\boxed{Torque = 14.87\,\text{N}\cdot\text{m}}$$

$$\text{Moment to Raise} = W\ r\ \tan(\theta + \phi_s)$$

$$\text{Moment to Lower} = W\ r\ \tan(\theta - \phi_s)$$

Journal Bearings. Axle Friction





- Journal bearings provide lateral support to rotating shafts. Thrust bearings provide axial support
- Frictional resistance of fully lubricated bearings depends on clearances, speed and lubricant viscosity. Partially lubricated axles and bearings can be assumed to be in direct contact along a straight line.
- Forces acting on bearing are weight *W* of wheels and shaft, couple *M* to maintain motion, and reaction *R* of the bearing.
- Reaction is vertical and equal in magnitude to W.
- Reaction line of action does not pass through shaft center *O*; *R* is located to the right of *O*, resulting in a moment that is balanced by *M*.
- Physically, contact point is displaced as axle "climbs" in bearing.

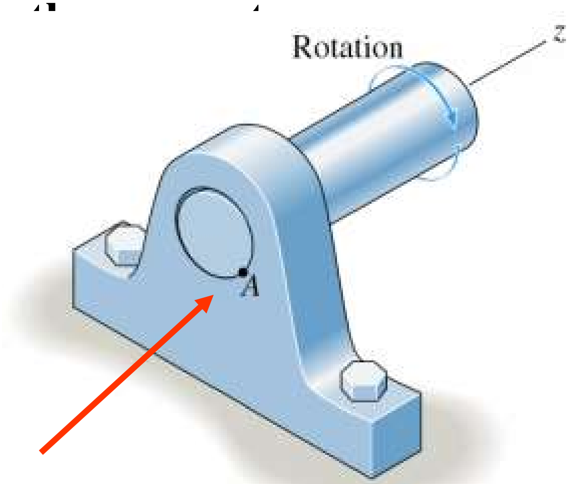
Journal Bearings. Axle Friction

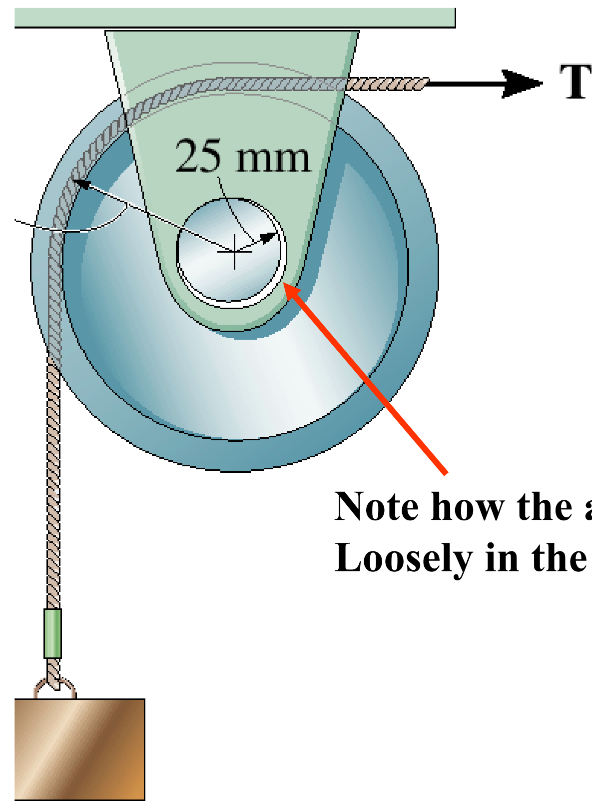
Axle Friction (Journal Bearings)

If the journal bearings fit the axle tightly and are well-lubricated, then the laws of fluid mechanics are used to determine the frictional resistance. However, if the bearing is somewhat loose fitting and is not well-lubricated, then the laws of dry friction apply. This is the approach used here.

Assumptions:

1) The axle is loose fitting
2) The bearing is not well-lubricated
3) As the axle rotates, it "climbs up" the wall of the support and there is a single point of contact between the axle and

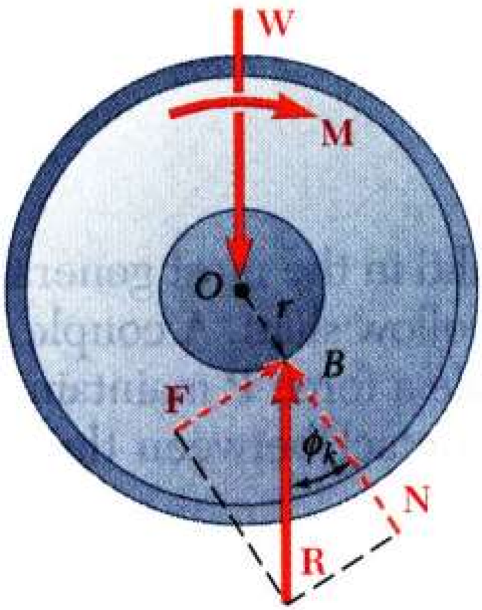




Note how the axle fits Loosely in the support.

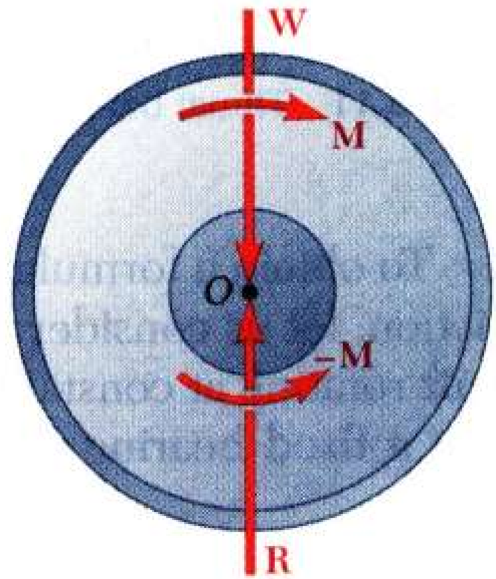
Note how the axle climbs up the wall of the support. A single point of contact, A, is assumed.

Journal Bearings. Axle Friction

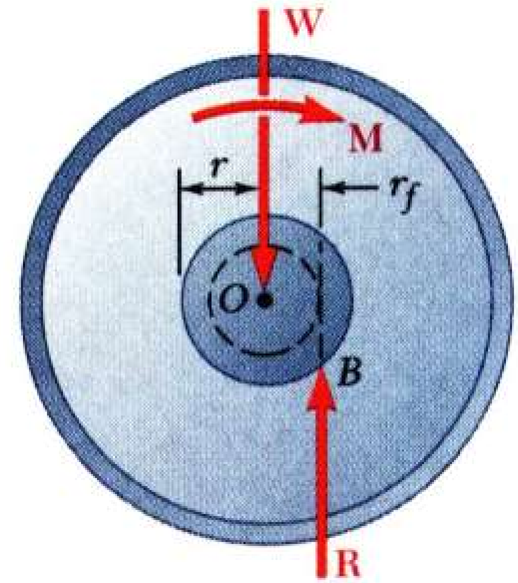


- Angle between R and normal to bearing surface is the angle of kinetic friction φ_k.

$$M = Rr\sin\phi_k$$
$$\approx Rr\mu_k$$



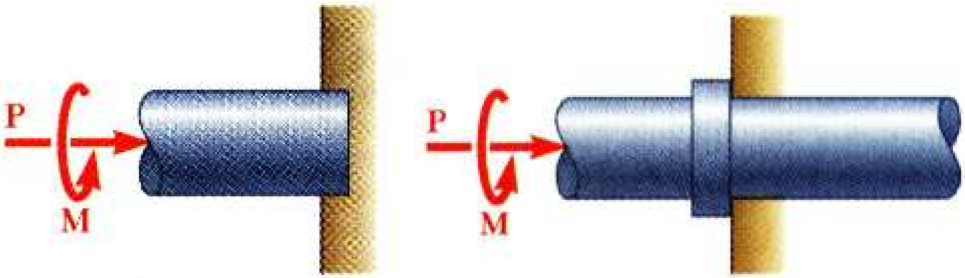
- May treat bearing reaction as force-couple system.



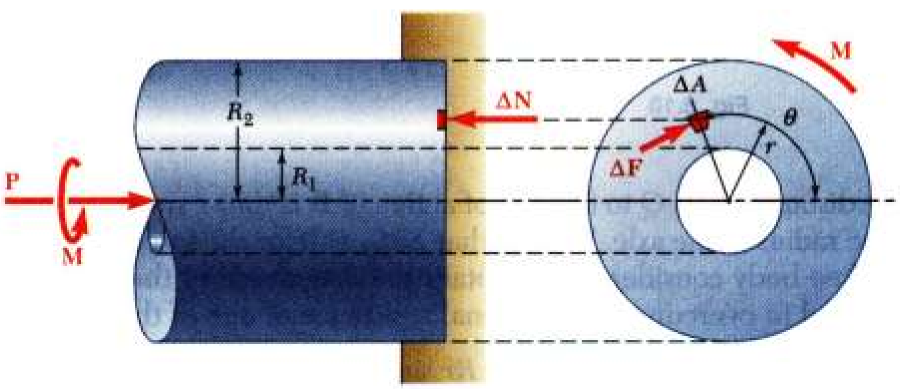
- For graphical solution, R must be tangent to *circle of friction*.

$$r_f = r\sin\phi_k$$
$$\approx r\mu_k$$

Thrust Bearings. Disk Friction



(a) End bearing (b) Collar bearing



Consider rotating hollow shaft:

$$\Delta M = r\Delta F = r\mu_k \Delta N = r\mu_k \frac{P}{A}\Delta A$$

$$= \frac{r\mu_k P\Delta A}{\pi\left(R_2^2 - R_1^2\right)}$$

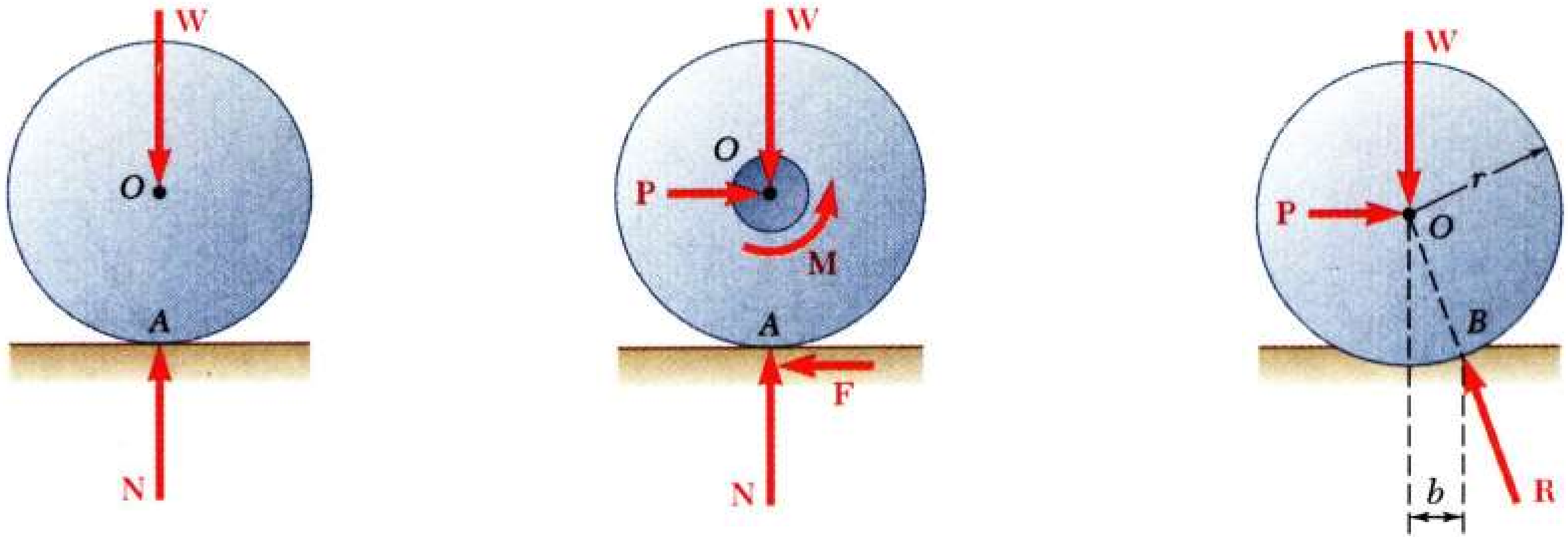
$$M = \frac{\mu_k P}{\pi\left(R_2^2 - R_1^2\right)} \int_0^{2\pi} \int_{R_1}^{R_2} r^2 dr d\theta$$

$$= \frac{2}{3}\mu_k P \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$$

For full circle of radius R,

$$M = \frac{2}{3}\mu_k PR$$

Wheel Friction. Rolling Resistance



- Point of wheel in contact with ground has no relative motion with respect to ground.

  Ideally, no friction.

- Moment M due to frictional resistance of axle bearing requires couple produced by equal and opposite P and F.

  Without friction at rim, wheel would slide.

- Deformations of wheel and ground cause resultant of ground reaction to be applied at B. P is required to balance moment of W about B.

  $Pr = Wb$

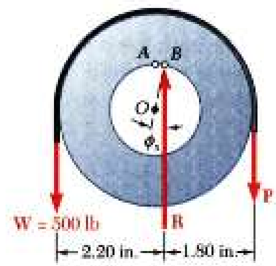
  b = coef of rolling resistance

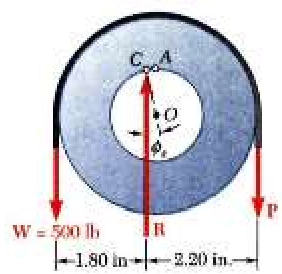
Sample Problem 8.6

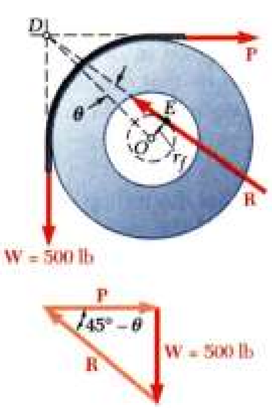
A pulley of diameter 4 in. can rotate about a fixed shaft of diameter 2 in. The coefficient of static friction between the pulley and shaft is 0.20.

Determine:

- the smallest vertical force *P* required to start raising a 500 lb load,
- the smallest vertical force *P* required to hold the load, and
- the smallest horizontal force P required to start raising the same load.



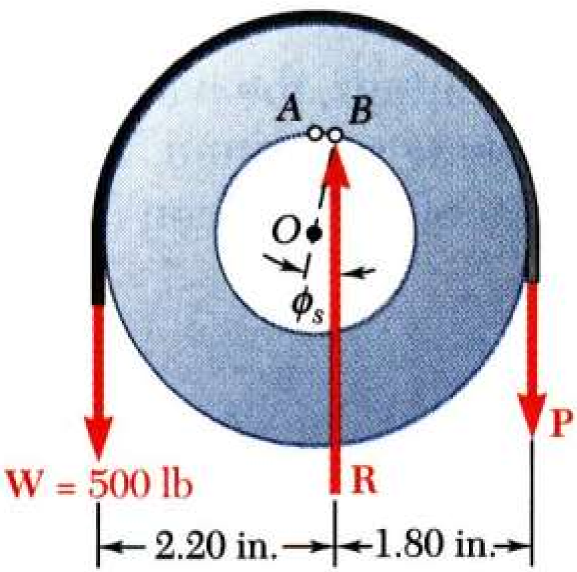




<u>SOLUTION</u>:

- With the load on the left and force *P* on the right, impending motion is clockwise to raise load. Sum moments about displaced contact point *B* to find *P*.
- Impending motion is counter-clockwise as load is held stationary with smallest force *P*. Sum moments about *C* to find *P*.
- With the load on the left and force *P* acting horizontally to the right, impending motion is clockwise to raise load. Utilize a force triangle to find *P*.

Sample Problem 8.6



SOLUTION:

- With the load on the left and force P on the right, impending motion is clockwise to raise load. Sum moments about displaced contact point B to find P.

  The perpendicular distance from center O of pulley to line of action of R is

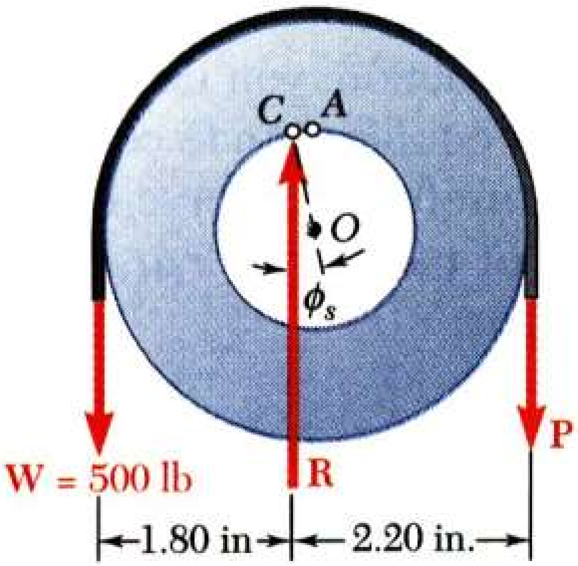
  $$r_f = r \sin \varphi_s \approx r\mu_s \quad r_f \approx (1\text{ in.})0.20 = 0.20\text{ in.}$$

  Summing moments about B,

  $$\sum M_B = 0: \qquad (2.20\text{ in.})(500\text{ lb}) - (1.80\text{ in.})P = 0$$

  $$\boxed{P = 611\text{ lb}}$$

Sample Problem 8.6



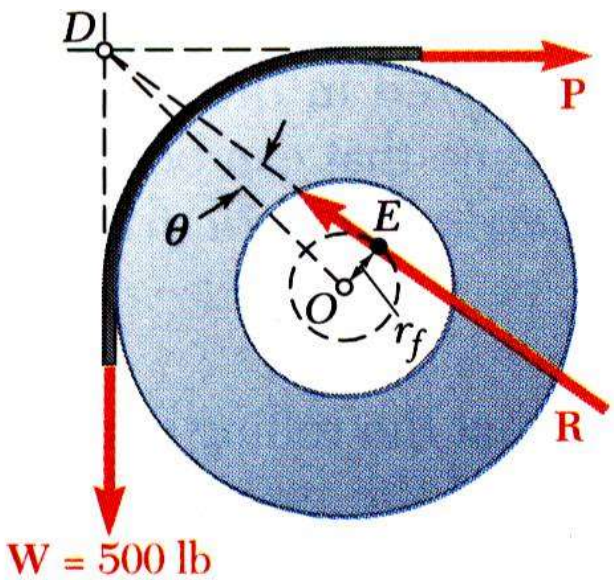
- Impending motion is counter-clockwise as load is held stationary with smallest force *P*. Sum moments about *C* to find *P*.

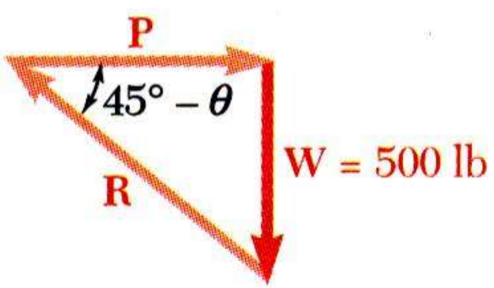
The perpendicular distance from center *O* of pulley to line of action of *R* is again 0.20 in. Summing moments about C,

$$\sum M_C = 0: \qquad (1.80\,\text{in.})(500\,\text{lb}) - (2.20\,\text{in.})P = 0$$

$$\boxed{P = 409\,\text{lb}}$$

Sample Problem 8.6





- With the load on the left and force *P* acting horizontally to the right, impending motion is clockwise to raise load. Utilize a force triangle to find *P*.

  Since *W*, *P*, and *R* are not parallel, they must be concurrent. Line of action of *R* must pass through intersection of *W* and *P* and be tangent to circle of friction which has radius $r_f = 0.20$ in.

$$\sin\theta = \frac{OE}{OD} = \frac{0.20\,\text{in.}}{(2\,\text{in.})\sqrt{2}} = 0.0707$$

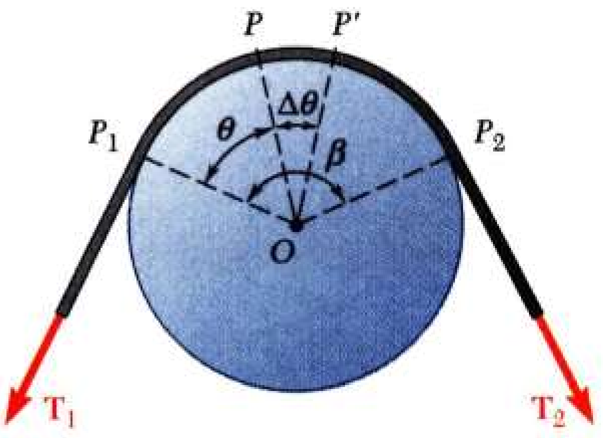
$$\theta = 4.1°$$

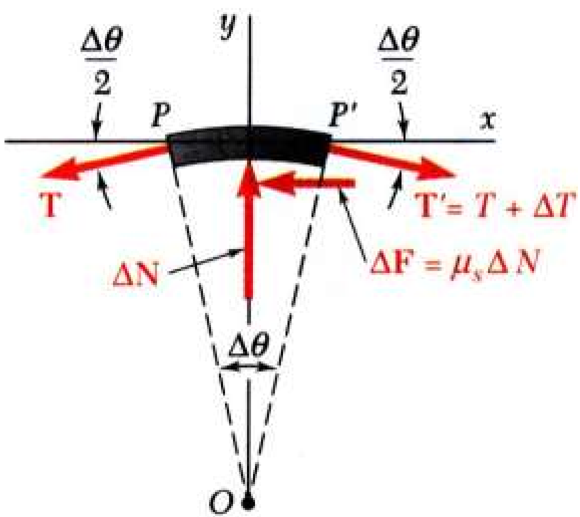
From the force triangle,

$$P = W\cot(45° - \theta) = (500\,\text{lb})\cot 40.9°$$

$$\boxed{P = 577\,\text{lb}}$$

Belt Friction





- Relate T_1 and T_2 when belt is about to slide to right.
- Draw free-body diagram for element of belt

$$\sum F_x = 0: \quad (T + \Delta T)\cos\frac{\Delta\theta}{2} - T\cos\frac{\Delta\theta}{2} - \mu_s \Delta N = 0$$

$$\sum F_y = 0: \quad \Delta N - (T + \Delta T)\sin\frac{\Delta\theta}{2} - T\sin\frac{\Delta\theta}{2} = 0$$

- Combine to eliminate ΔN, divide through by $\Delta\theta$,

$$\frac{\Delta T}{\Delta\theta}\cos\frac{\Delta\theta}{2} - \mu_s\left(T + \frac{\Delta T}{2}\right)\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}$$

- In the limit as $\Delta\theta$ goes to zero,

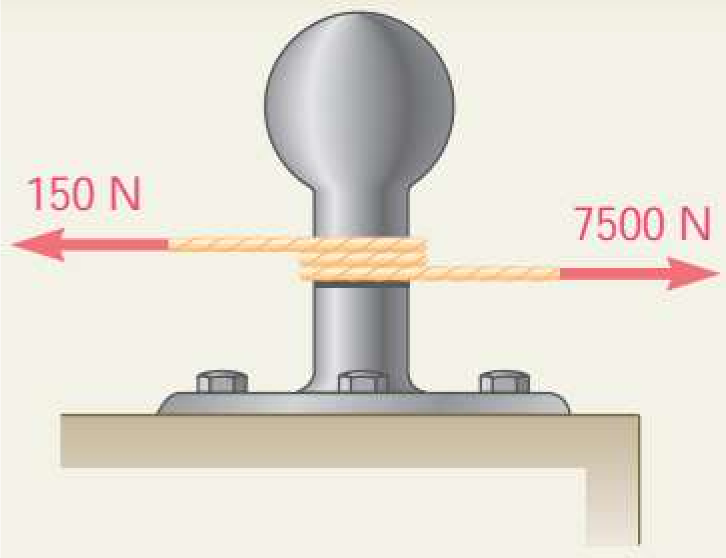
$$\frac{dT}{d\theta} - \mu_s T = 0$$

- Separate variables and integrate from $\theta = 0$ to $\theta = \beta$

$$\ln\frac{T_2}{T_1} = \mu_s \beta \quad \text{or} \quad \frac{T_2}{T_1} = e^{\mu_s \beta}$$

Sample Problem 8.7

A hawser thrown from a ship to a pier is wrapped two full turns around a bollard. The tension in the hawser is 7500 N; by exerting a force of 150 N on its free end, a dockworker can just keep the hawser from slipping. (*a*) Determine the coefficient of friction between the hawser and the bollard. (*b*) Determine the tension in the hawser that could be resisted by the 150-N force if the hawser were wrapped three full turns around the bollard.



Sample Problem 8.7

a. Coefficient of Friction. Since slipping of the hawser is impending, we use Eq. (8.13):

$$\ln \frac{T_2}{T_1} = m_s b$$

Since the hawser is wrapped two full turns around the bollard, we have

$$b = 2(2p \text{ rad}) = 12.57 \text{ rad}$$

$$T_1 = 150 \text{ N} \qquad T_2 = 7500 \text{ N}$$

Therefore,

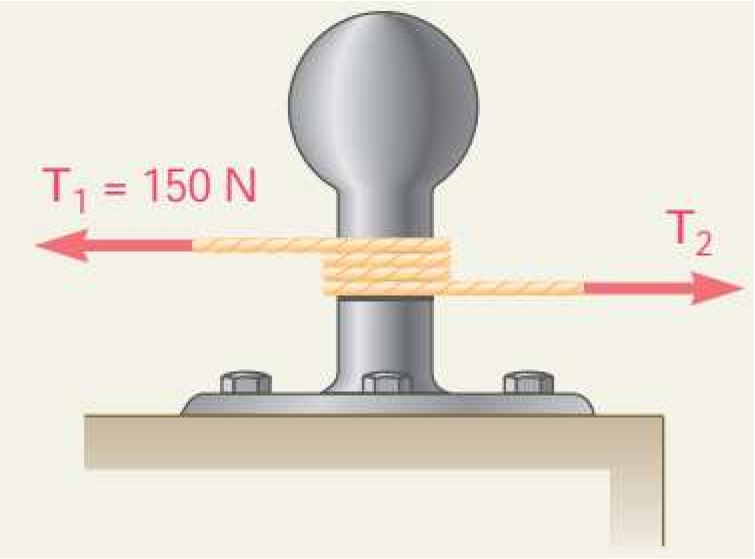
$$m_s b = \ln \frac{T_2}{T_1}$$

$$m_s(12.57 \text{ rad}) = \ln \frac{7500 \text{ N}}{150 \text{ N}} = \ln 50 = 3.91$$

$$m_s = 0.311$$

$$m_s = 0.311$$

Sample Problem 8.7



Sample Problem 8.7

b. Hawser Wrapped Three Turns Around Bollard. Using the value of μ_s obtained in part *a*, we now have

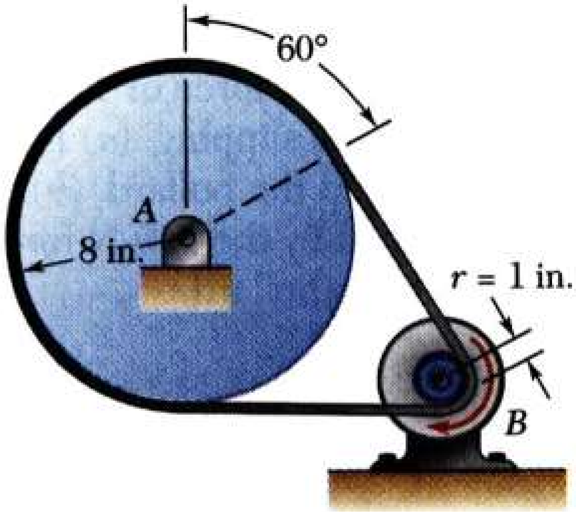
$$\beta = 3(2\pi \text{ rad}) = 18.85 \text{ rad}$$
$$T_1 = 150 \text{ N} \qquad \mu_s = 0.311$$

Substituting these values into Eq. (8.14), we obtain

$$\frac{T_2}{T_1} = e^{\mu_s \beta}$$
$$\frac{T_2}{150 \text{ N}} = e^{(0.311)(18.85)} = e^{5.862} = 351.5$$
$$T_2 = 52\,725 \text{ N}$$

$$T_2 = 52.7 \text{ kN}$$

Sample Problem 8.8



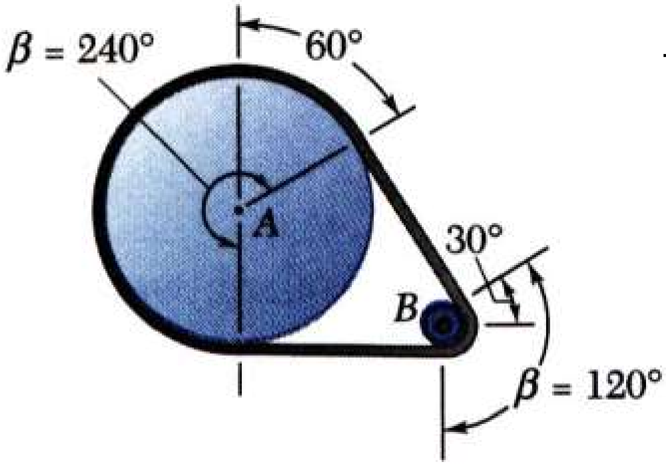
A flat belt connects pulley *A* to pulley *B*. The coefficients of friction are $\mu_s = 0.25$ and $\mu_k = 0.20$ between both pulleys and the belt.

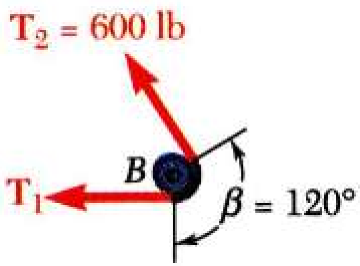
Knowing that the maximum allowable tension in the belt is 600 lb, determine the largest torque which can be exerted by the belt on pulley *A*.

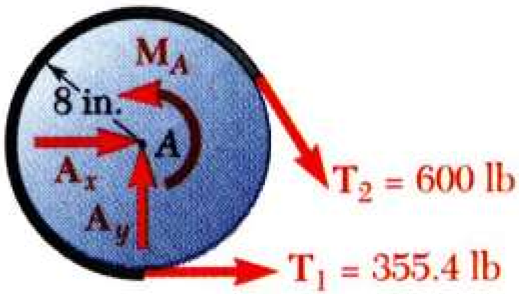
SOLUTION:

- Since angle of contact is smaller, slippage will occur on pulley *B* first. Determine belt tensions based on pulley *B*.
- Taking pulley A as a free-body, sum moments about pulley center to determine torque.

Sample Problem 8.8







SOLUTION:

- Since angle of contact is smaller, slippage will occur on pulley *B* first. Determine belt tensions based on pulley *B*.

$$\frac{T_2}{T_1} = e^{\mu_s \beta} \qquad \frac{600\,\text{lb}}{T_1} = e^{0.25(2\pi/3)} = 1.688$$

$$T_1 = \frac{600\,\text{lb}}{1.688} = 355.4\,\text{lb}$$

- Taking pulley *A* as free-body, sum moments about pulley center to determine torque.

$$\sum M_A = 0: \qquad M_A + (8\,\text{in.})(355.4\,\text{lb} - 600\,\text{lb}) = 0$$

$$\boxed{M_A = 163.1\,\text{lb}\cdot\text{ft}}$$

CHAPTER

9

VECTOR MECHANICS FOR ENGINEERS:

STATICS

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Distributed Forces:
Moments of Inertia

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Introduction

The strength of structural members depends to a large extent on the properties of their cross sections, particularly on the second moments, or moments of inertia, of their areas.

Introduction

Chapter 9 – Moments of Inertia

Moments of inertia are not actually used in Statics; however, since the calculations for moments of inertia are quite similar to those for centroids they are introduced at this point. Moments of inertia are used in courses such as mechanics of materials, dynamics, and fluid mechanics.

The ***moment of inertia*** of an object is a measure of its resistance to change in rotation. Everyday experience tells us that it is harder to start (or stop) a large wheel turning than a small wheel. Mathematically, this is represented by the large wheel having a larger moment of inertia.

Moments of inertia are used in various engineering calculations, including:

- Locating the resultant of hydrostatic pressure forces on submerged bodies
- Calculating stresses in beams – they are at times related to the moment of inertia of the cross-sectional area of the beam (resistance to bending)
- Mass moments of inertia are used in studying the rotational motion of objects

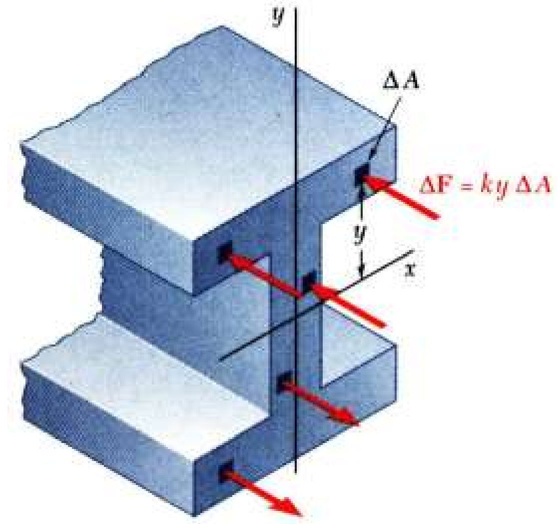
Introduction

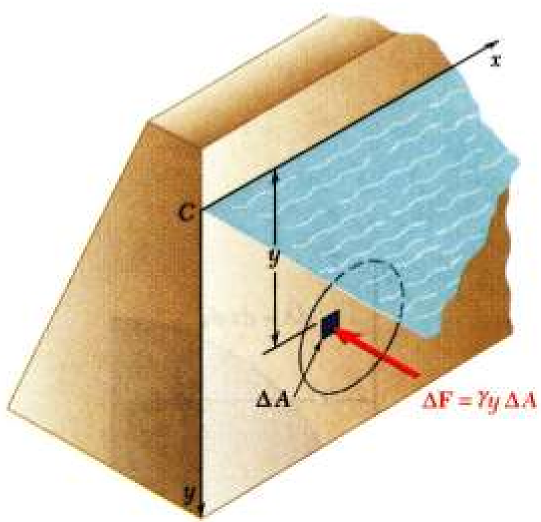
- Previously considered distributed forces which were proportional to the area or volume over which they act.
  - The resultant was obtained by summing or integrating over the areas or volumes.
  - The moment of the resultant about any axis was determined by computing the first moments of the areas or volumes about that axis.

- Will now consider forces which are proportional to the area or volume over which they act but also vary linearly with distance from a given axis.
  - It will be shown that the magnitude of the resultant depends on the first moment of the force distribution with respect to the axis.
  - The point of application of the resultant depends on the second moment of the distribution with respect to the axis.

- Current chapter will present methods for computing the moments and products of inertia for areas and masses.

Moment of Inertia of an Area





- Consider distributed forces $\Delta\vec{F}$ whose magnitudes are proportional to the elemental areas ΔA on which they act and also vary linearly with the distance of ΔA from a given axis.

- Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta\vec{F} = ky\Delta A$$

$$R = k\int y\,dA = 0 \qquad \int y\,dA = Q_x = \text{first moment}$$

$$M = k\int y^2 dA \qquad \int y^2 dA = \text{second moment}$$

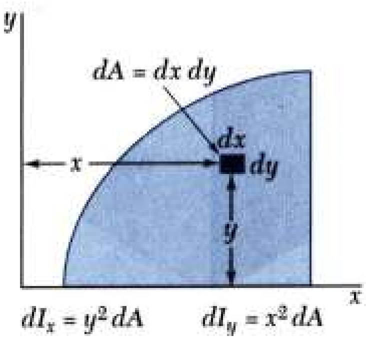
- Example: Consider the net hydrostatic force on a submerged circular gate.

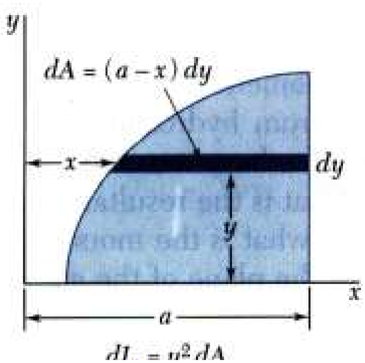
$$\Delta F = p\Delta A = \gamma y\Delta A$$

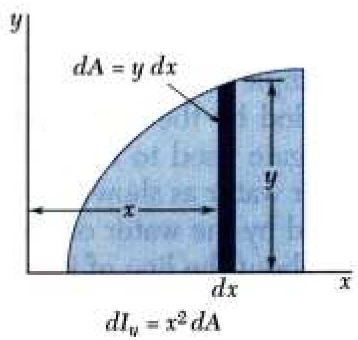
$$R = \gamma\int y\,dA$$

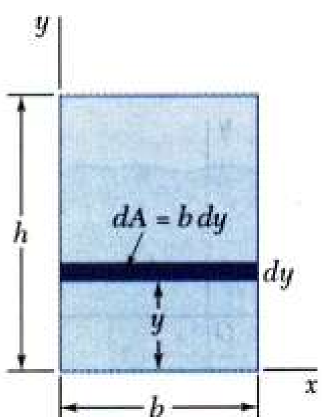
$$M_x = \gamma\int y^2 dA$$

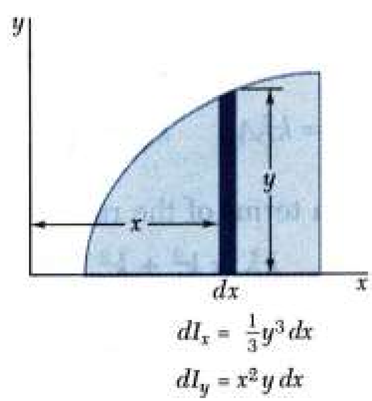
Moment of Inertia of an Area by Integration











- *Second moments* or *moments of inertia* of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

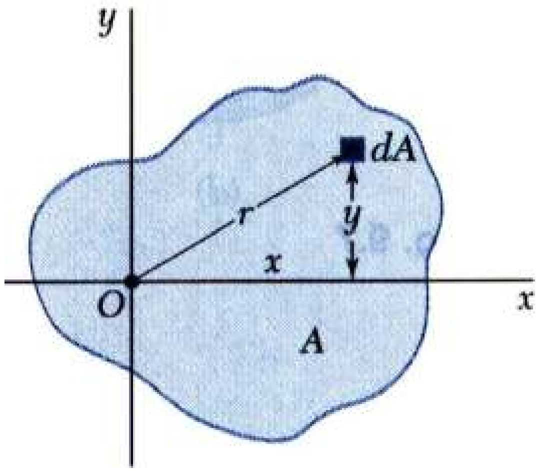
- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \tfrac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\,dx$$

Polar Moment of Inertia



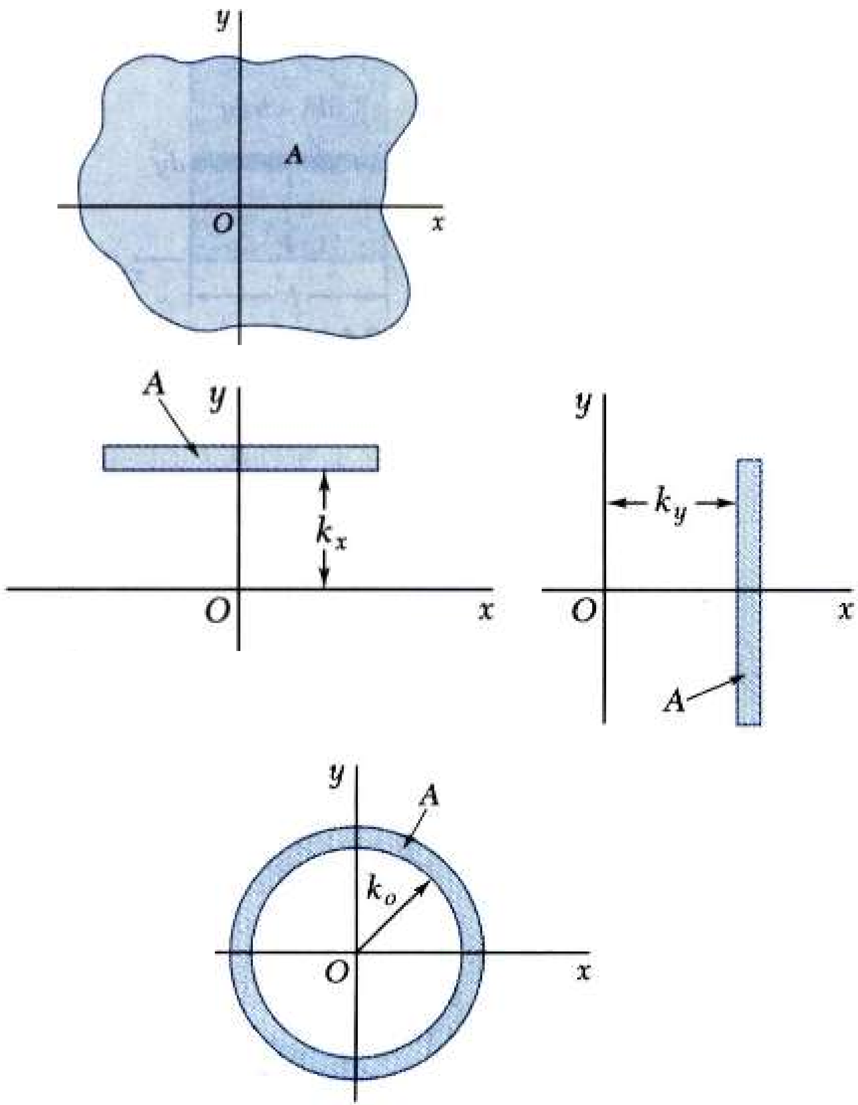
- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = \int r^2 dA$$

- The polar moment of inertia is related to the rectangular moments of inertia,

$$J_0 = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Radius of Gyration of an Area



- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$

Determining moments of inertia

Moments of inertia can be found using three methods:

1. Composites – If an object can be divided up into relatively simple shapes with known moments of inertia, then the moment of inertia of the entire is the sum of the moments of inertia of the composites. For N composite shapes:

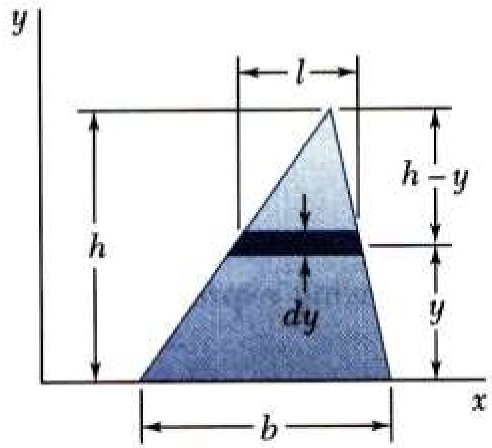
$$\boxed{\begin{aligned} I_x &= I_{x_1} + I_{x_2} + \ldots + I_{x_N} \\ I_y &= I_{y_1} + I_{y_2} + \ldots + I_{y_N} \end{aligned}}$$

2. Integration – If the area, volume, or line of an object can be described by a mathematical equations, then the moment of inertia can be determined through integration.

$$\boxed{I_x = \int y^2 dA \qquad I_y = \int x^2 dA}$$

3. Solid modeling software – Software such as AutoCAD can be used to construct 3D models of objects. The software can also determine the centroid , volumes, moments of inertia and other mass properties of the object). This is not a required element of this course, but an example will be provided.

Sample Problem 9.1



Determine the moment of inertia of a triangle with respect to its base.

SOLUTION:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

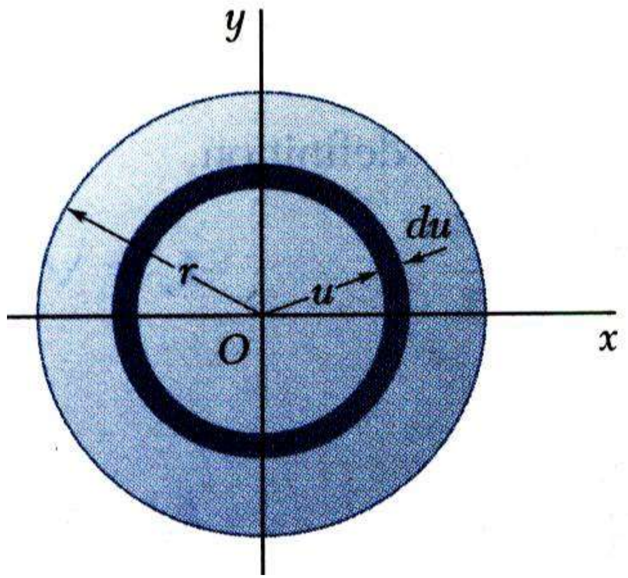
- Integrating dI_x from $y = 0$ to $y = h$,

$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$

Sample Problem 9.2



a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

b) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.

SOLUTION:

- An annular differential area element is chosen,

$$dJ_O = u^2 dA \qquad dA = 2\pi u\, du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du$$

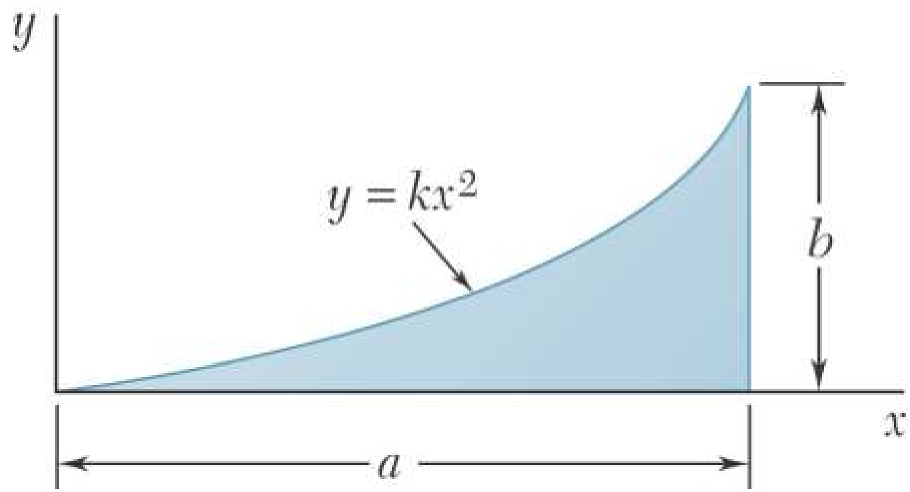
$$\boxed{J_O = \frac{\pi}{2} r^4}$$

- From symmetry, $I_x = I_y$,

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2} r^4 = 2I_x$$

$$\boxed{I_{diameter} = I_x = \frac{\pi}{4} r^4}$$

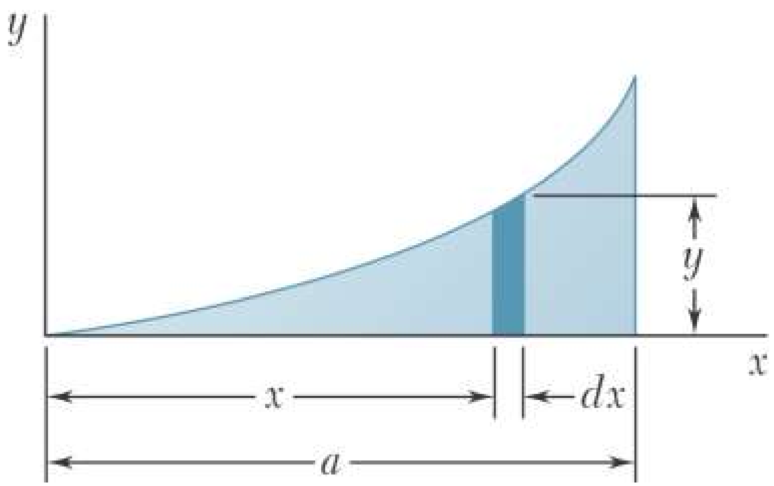
Sample Problem 9.3



a) Determine the moment of inertia of the shaded area shown with respect to the coordinate axes

b) Using the result of part *a*, determine the radius of gyration of the shaded area with respect to each of the cordiante axes

Sample Problem 9.3



$$y = \frac{b}{a^2}x^2, \quad A = \frac{1}{3}ab$$

$$dI_x = \frac{1}{3}y^3dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx$$

$$I_x = \int_0^a \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\left(\frac{b^3}{a^6}\frac{x^7}{7}\right)_0^a$$

$$I_x = \frac{ab^3}{21}$$

$$dI_y = x^2dA = x^2ydx = x^2\left(\frac{b}{a^2}x^2\right)dx$$

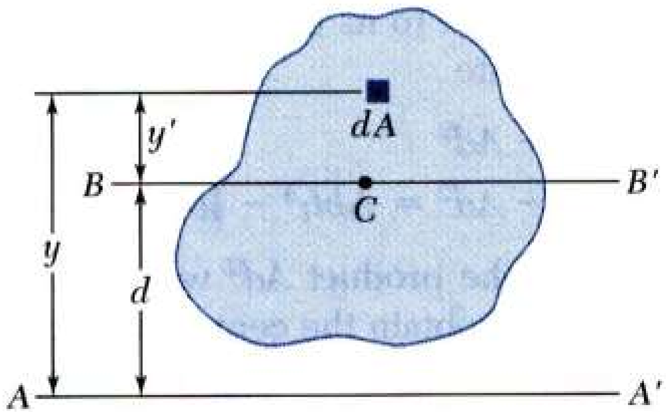
$$I_y = \int_0^a \frac{b}{a^2}x^4dx = \frac{b}{a^2}\frac{x^5}{5}\Big]_0^a$$

$$I_y = \frac{a^3b}{5}$$

$$k_x^2 = \frac{I_x}{A} = \frac{b^2}{7}$$

$$k_y^2 = \frac{I_y}{A} = \frac{3a^2}{5}$$

Parallel Axis Theorem



- Consider moment of inertia *I* of an area *A* with respect to the axis *AA'*

$$I = \int y^2 dA$$

- The axis *BB'* passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I = \bar{I} + Ad^2 \qquad \textit{parallel axis theorem}$$

Parallel Axis Theorem

Parallel-axis theorem

Changing the axis of rotation can have a huge effect on the moment of inertia. Consider the case of trying to rotate a wheel about its axle (its centroidal axis) versus the much more difficult task of trying to rotate the wheel about a bar (axis) several feet away from the wheel. The ***parallel axis theorem*** is used to calculate the moment of inertia about a new axis if the moment of inertia about the centroidal axis is known.

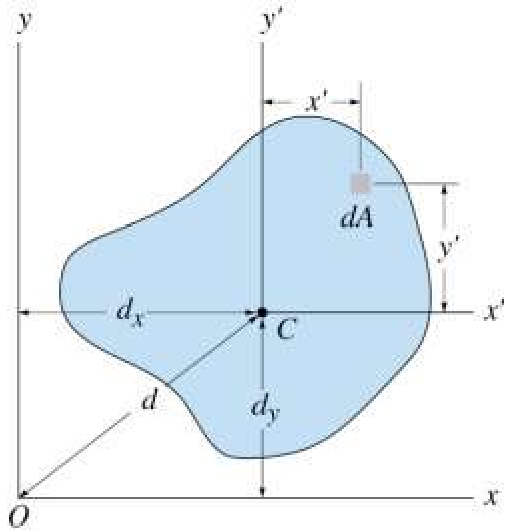


Figure 10.03

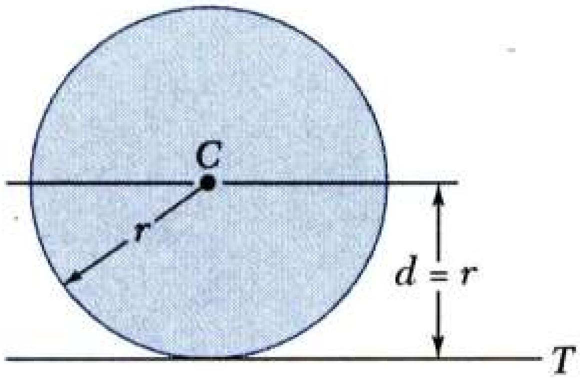
$$I_y = \int x^2 dA = \int (x' + dx)^2 dA$$
$$= \int (x')^2 dA + 2dx \int x'dA + dx^2 \int dA$$
$$= I_y' + dx^2 \cdot A$$

Parallel-axis theorem:

$$I_x = I_x' + dy^2 \cdot A$$
$$I_y = I_y' + dx^2 \cdot A$$
$$J_o = J_o' + d^2 \cdot A, \quad \text{where } d^2 = dx^2 + dy^2$$

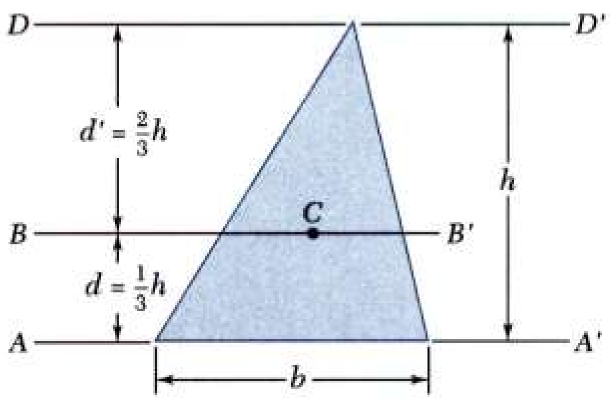
Important note: The parallel axis theorem shows how to find the moment of inertia about any other axis if the moment of inertia is known about the centroidal axis. So you must begin by finding the moment of inertia about the **centroidal axis** (i. e., find I_x ' first to find I_x about any other axis).

Parallel Axis Theorem



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right) r^2$$
$$= \tfrac{5}{4}\pi r^4$$

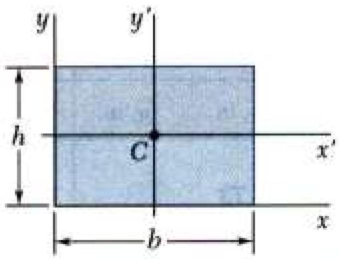
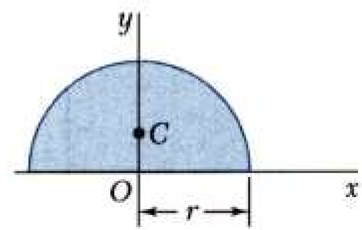
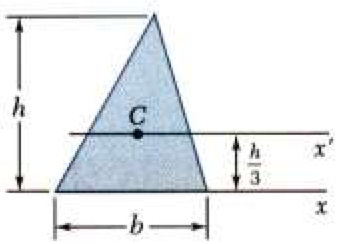
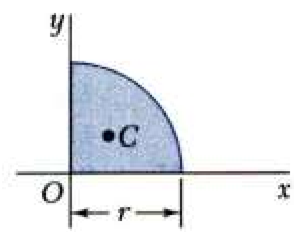
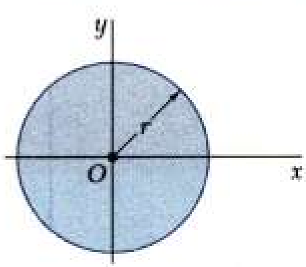
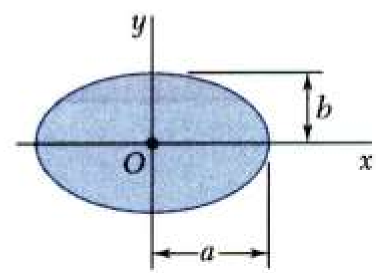


- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12} bh^3 - \tfrac{1}{2} bh\left(\tfrac{1}{3} h\right)^2$$
$$= \tfrac{1}{36} bh^3$$

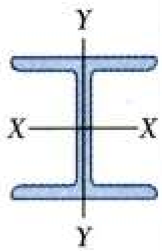
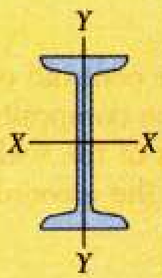
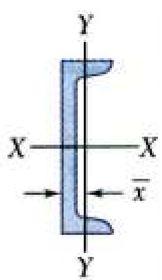
Moments of Inertia of Composite Areas

- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

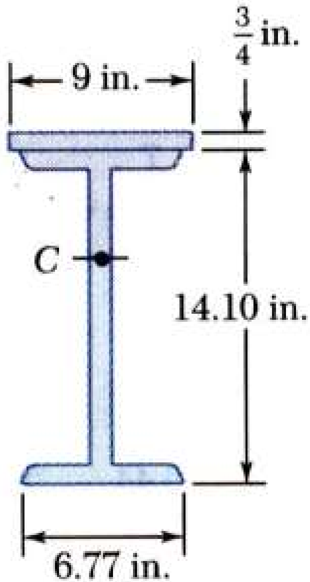
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$	Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Moments of Inertia of Composite Areas

Moments of Inertia of Composite Areas

	Designation	Area mm^2	Depth mm	Width mm	Axis X-X $\bar{I}_x$ 10^6 mm^4	Axis X-X $\bar{k}_x$ mm	Axis X-X $\bar{y}$ mm	Axis Y-Y $\bar{I}_y$ 10^6 mm^4	Axis Y-Y $\bar{k}_y$ mm	Axis Y-Y $\bar{x}$ mm
W Shapes (Wide-Flange Shapes) 	W460 × 113†	14400	463	280	554	196.3		63.3	66.3	
	W410 × 85	10800	417	181	316	170.7		17.94	40.6	
	W360 × 57	7230	358	172	160.2	149.4		11.11	39.4	
	W200 × 46.1	5890	203	203	45.8	88.1		15.44	51.3	
S Shapes (American Standard Shapes) 	S460 × 81.4†	10390	457	152	335	179.6		8.66	29.0	
	S310 × 47.3	6032	305	127	90.7	122.7		3.90	25.4	
	S250 × 37.8	4806	254	118	51.6	103.4		2.83	24.2	
	S150 × 18.6	2362	152	84	9.2	62.2		0.758	17.91	
C Shapes (American Standard Channels) 	C310 × 30.8†	3929	305	74	53.7	117.1		1.615	20.29	17.73
	C250 × 22.8	2897	254	65	28.1	98.3		0.949	18.11	16.10
	C200 × 17.1	2181	203	57	13.57	79.0		0.549	15.88	14.50
	C150 × 12.2	1548	152	48	5.45	59.4		0.288	13.64	13.00

Sample Problem 9.4



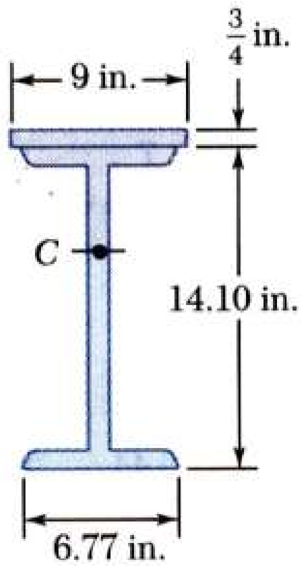
The strength of a W14x38 rolled steel beam is increased by attaching a plate to its upper flange.

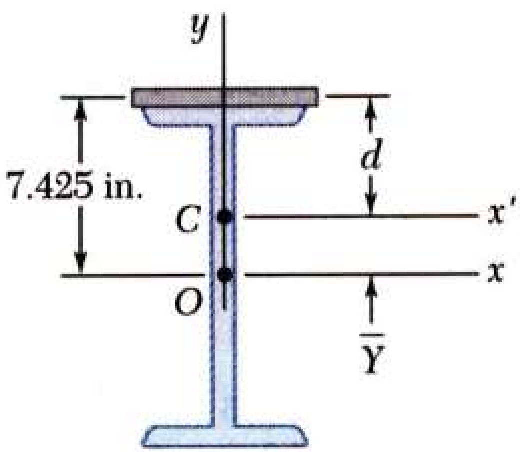
Determine the moment of inertia and radius of gyration with respect to an axis which is parallel to the plate and passes through the centroid of the section.

SOLUTION:

- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.
- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.
- Calculate the radius of gyration from the moment of inertia of the composite section.

Sample Problem 9.4





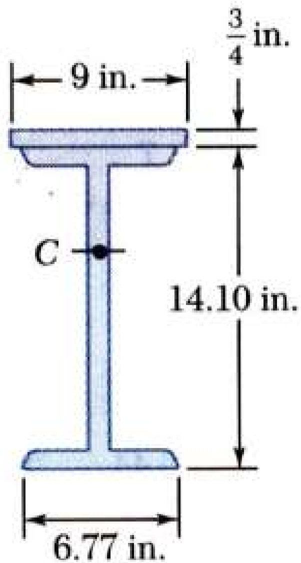
SOLUTION:

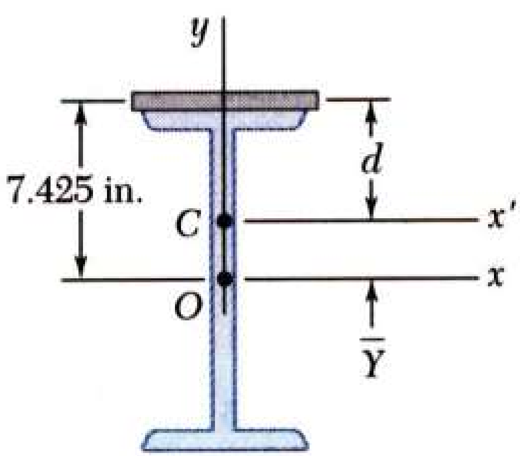
- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.

Section	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
Plate	6.75	7.425	50.12
Beam Section	11.20	0	0
	$\sum A = 17.95$		$\sum \bar{y}A = 50.12$

$$\bar{Y}\sum A = \sum \bar{y}A \qquad \bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{50.12\text{ in}^3}{17.95\text{ in}^2} = 2.792\text{ in.}$$

Sample Problem 9.4





- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

$$I_{x',\text{beam section}} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.20)(2.792)^2$$
$$= 472.3\ \text{in}^4$$

$$I_{x',\text{plate}} = \bar{I}_x + Ad^2 = \tfrac{1}{12}(9)\left(\tfrac{3}{4}\right)^3 + (6.75)(7.425 - 2.792)^2$$
$$= 145.2\ \text{in}^4$$

$$I_{x'} = I_{x',\text{beam section}} + I_{x',\text{plate}} = 472.3 + 145.2$$

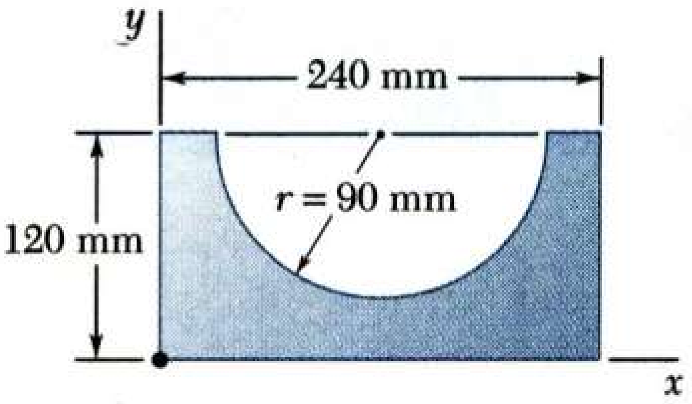
$$\boxed{I_{x'} = 618\ \text{in}^4}$$

- Calculate the radius of gyration from the moment of inertia of the composite section.

$$k_{x'} = \sqrt{\frac{I_{x'}}{A}} = \frac{617.5\ \text{in}^4}{17.95\ \text{in}^2}$$

$$\boxed{k_{x'} = 5.87\ \text{in.}}$$

Sample Problem 9.5

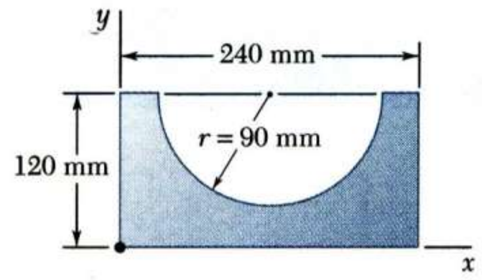


Determine the moment of inertia of the shaded area with respect to the x axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the x axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

Sample Problem 9.5

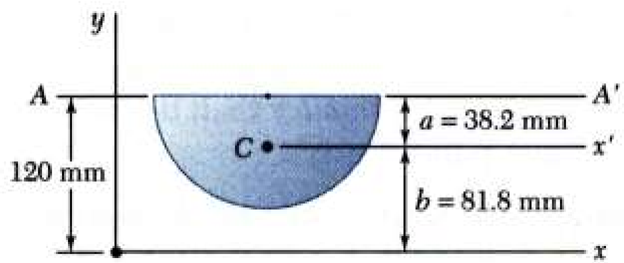


SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the x axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120) = 138.2 \times 10^6 \text{mm}^4$$



$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2 \text{ mm}$$

$$\text{b} = 120 - \text{a} = 81.8 \text{ mm}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$
$$= 12.72 \times 10^3 \text{mm}^2$$

Half-circle:

moment of inertia with respect to AA',

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76 \times 10^6 \text{mm}^4$$

moment of inertia with respect to x',

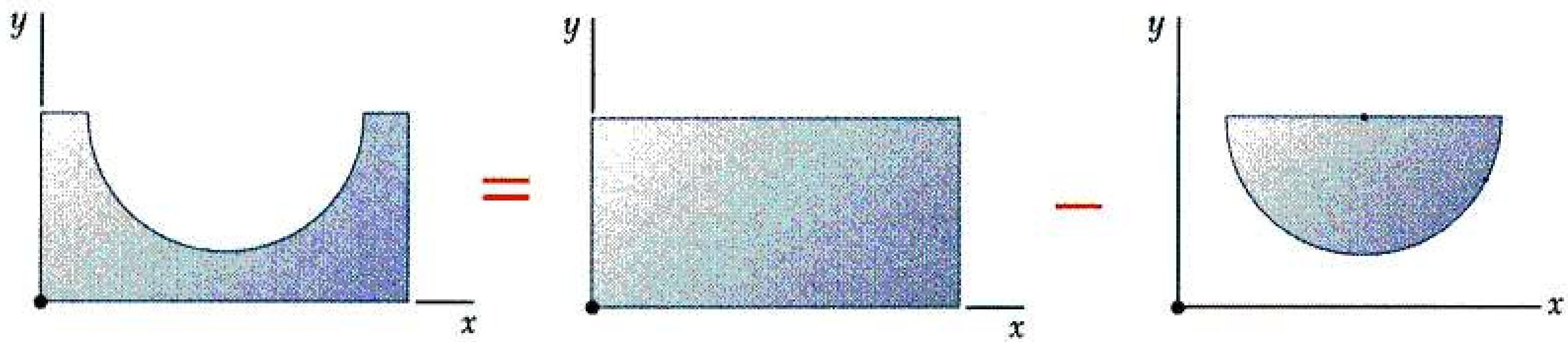
$$\bar{I}_{x'} = I_{AA'} - Aa^2 = (25.76 \times 10^6)(12.72 \times 10^3)$$
$$= 7.20 \times 10^6 \text{mm}^4$$

moment of inertia with respect to x,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 + (12.72 \times 10^3)(81.8)^2$$
$$= 92.3 \times 10^6 \text{mm}^4$$

Sample Problem 9.5

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.



$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

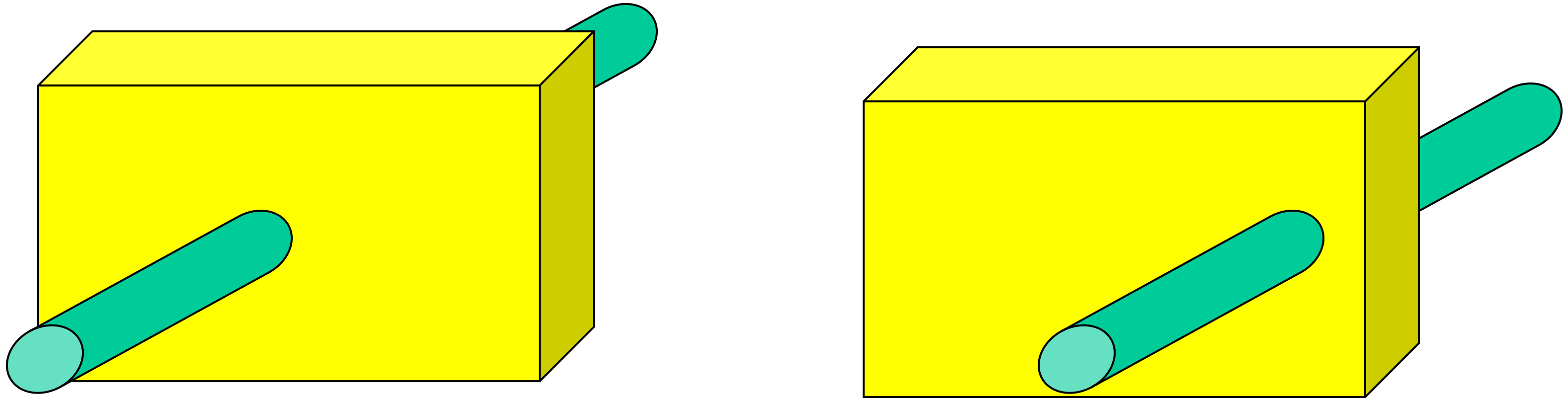
$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

Demonstration: Moments of inertia

<u>Demonstration:</u> The instructor will pass around two blocks of wood with dowels through them. Rotate the blocks using the dowels.

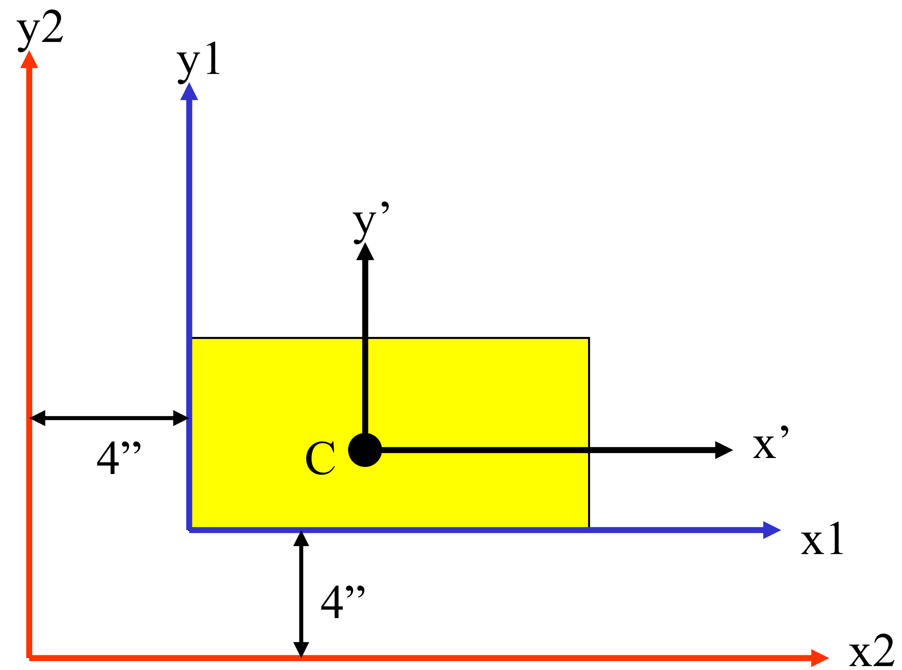
1) Which is easiest to rotate?

2) Where do you think that the dowel should be placed such that the block will be easiest to rotate? Why?



Example: (Also see Sample Problems 9.4 – 9.5)

Example: Find I_x and I_y for a 3.5” x 7” rectangular block
a) about the centroidal axes, x’ and y’ (refer to the table of composites)
b) about the axes along the edge of the block, x1 and y1
c) about the axes that are 4” away from the edge of the block, x2 and y2
Discuss the relative magnitudes of the results.



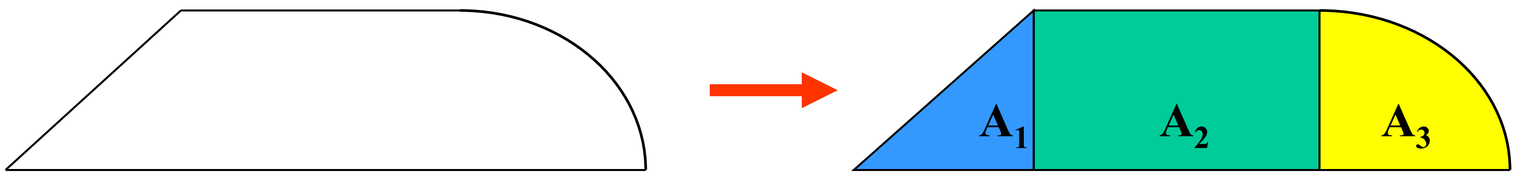
Moments of Inertia using composites

Finding moments of inertia using composites:

Moments of inertia are additive. If an object can be broken down into N composite shapes with known moments of inertia, then the total moment of inertia is the sum of the moments of inertia for each part. Also note that I_x and I_y are negative for negative areas.

Example:

The object below can be broken up into three areas with common shapes as shown.



If

I_{x1} and I_{y1} = the moments of inertia for area A_1

I_{x2} and I_{y2} = the moments of inertia for area A_2

I_{x3} and I_{y3} = the moments of inertia for area A_3

then

$$I_x(\text{total}) = \Sigma I_x = I_{x1} + I_{x2} + I_{x3}$$

$$I_y(\text{total}) = \Sigma I_y = I_{y1} + I_{y2} + I_{y3}$$

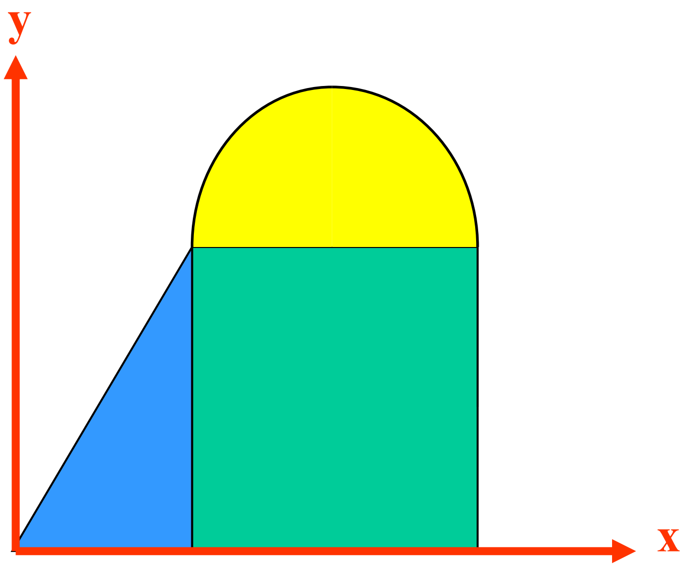
Moments of Inertia using composites

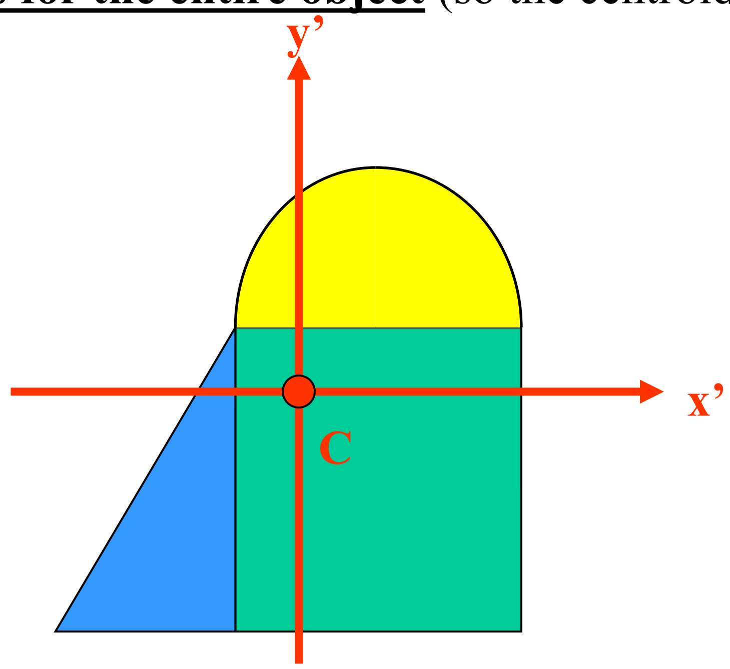
Finding moments of inertia using composites:

When moments of inertia are found using composites, two types of problems are typically considered:

1) Finding I_x and I_y around **fixed axes** (or specified axes).

2) Finding I_x and I_y around **the centroidal axes for the entire object** (so the centroid must be found first).



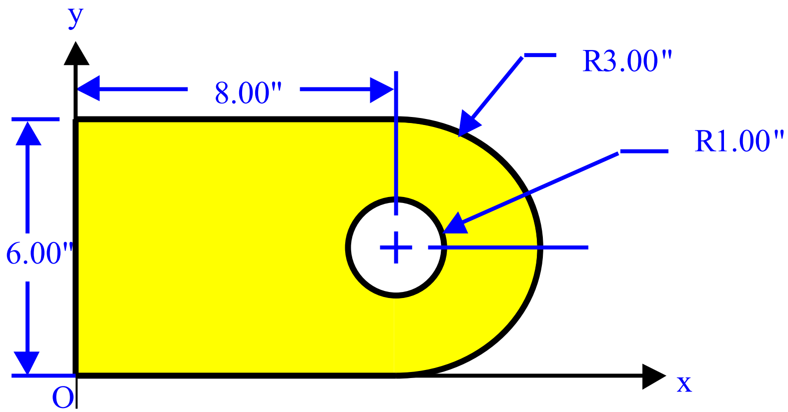


Example: Find I_x and I_y about the fixed x and y axes shown.

Example: Find I_x and I_y about the centroidal axes of the object.

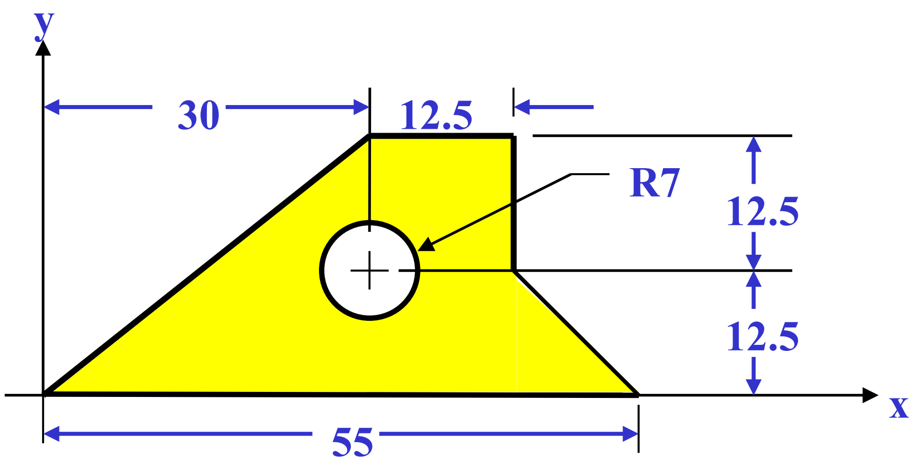
Finding moments of inertia using composites and fixed axes.

Example: Determine the moments of inertia about the ***fixed x and y axes*** shown.

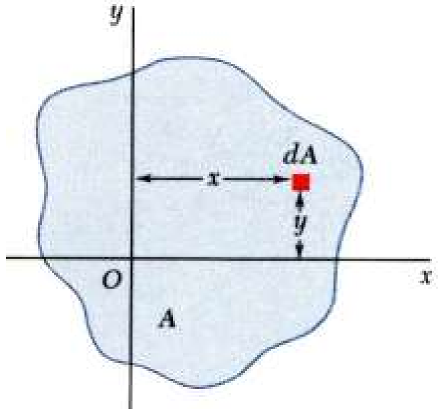


Finding moments of inertia using composites and centroidal axes.

Example: Determine the moments of inertia about the ***centroidal axes*** (i.e., about the centroid of the entire object). All dimensions are in mm.



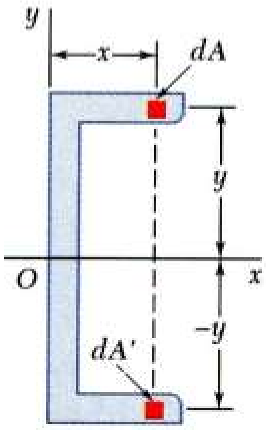
Product of Inertia

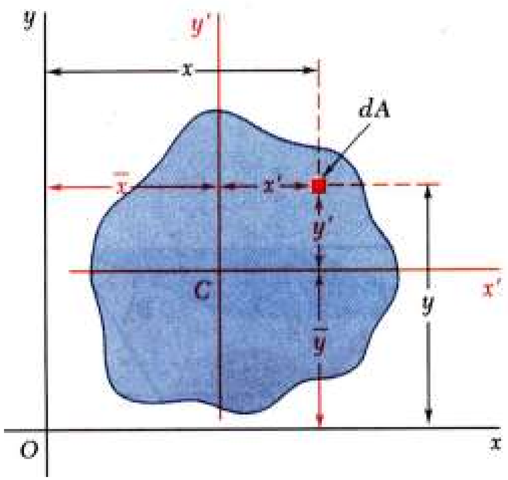


- *Product of Inertia:*

$$I_{xy} = \int xy\, dA$$

- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.

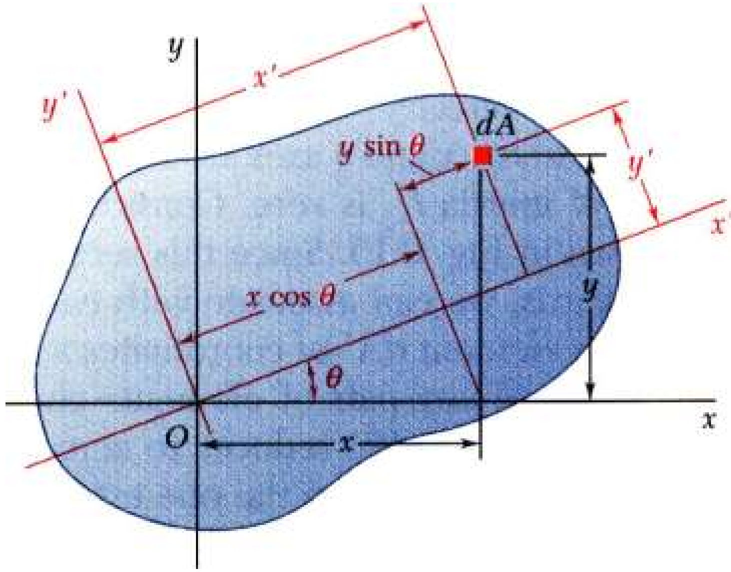




- Parallel axis theorem for products of inertia:

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$

Principal Axes and Principal Moments of Inertia



Given $I_x = \int y^2 dA \quad I_y = \int x^2 dA$

$$I_{xy} = \int xy\, dA$$

we wish to determine moments and product of inertia with respect to new axes x' and y'.

Note: $x' = x\cos\theta + y\sin\theta$

$$y' = y\cos\theta - x\sin\theta$$

- The change of axes yields

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

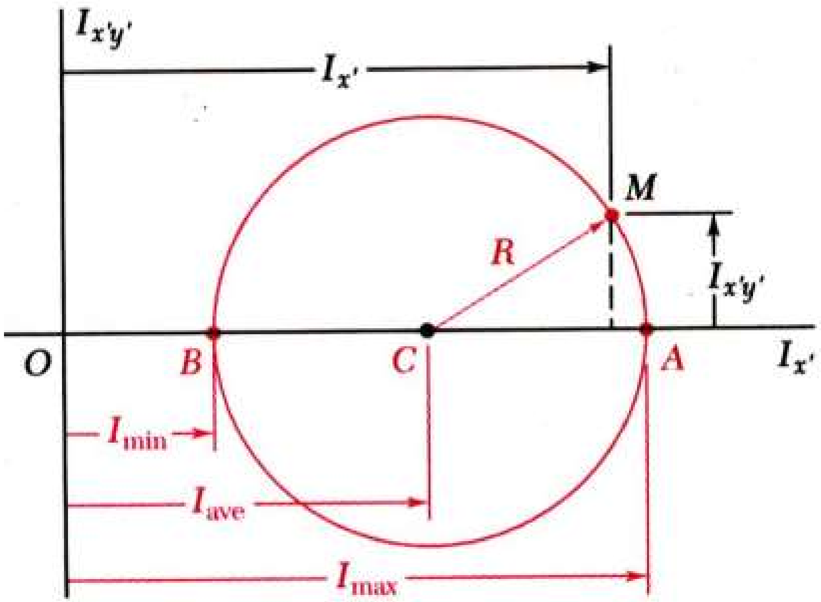
- The equations for $I_{x'}$ and $I_{x'y'}$ are the parametric equations for a circle,

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right) + I_{xy}^2}$$

- The equations for $I_{y'}$ and $I_{x'y'}$ lead to the same circle.

Principal Axes and Principal Moments of Inertia



$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right) + I_{xy}^2}$$

- At the points *A* and *B*, $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively.

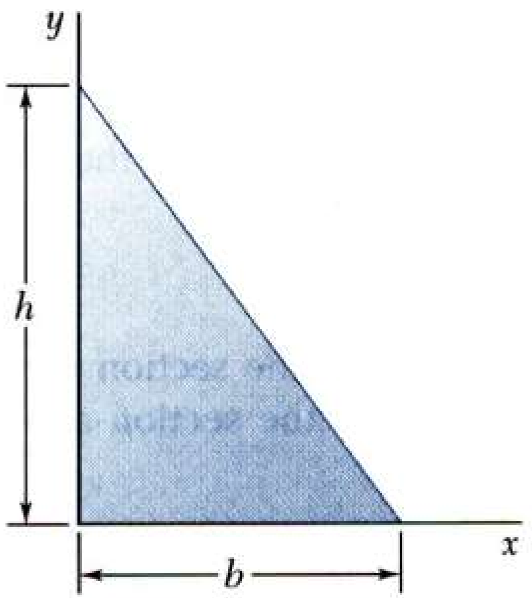
$$I_{\max,\min} = I_{ave} \pm R$$

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

- The equation for Θ_m defines two angles, 90° apart which correspond to the *principal axes* of the area about *O*.

- I_{max} and I_{min} are the *principal moments of inertia* of the area about *O*.

Sample Problem 9.6

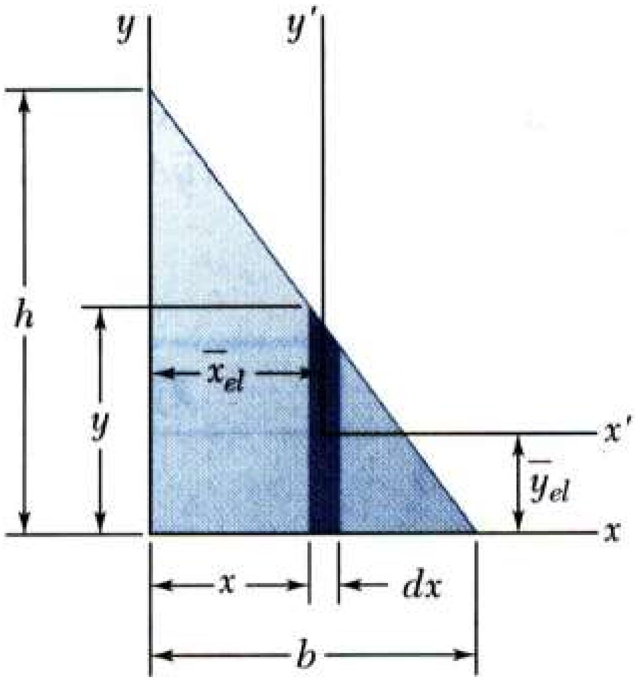


Determine the product of inertia of the right triangle *(a)* with respect to the x and y axes and *(b)* with respect to centroidal axes parallel to the x and y axes.

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

Sample Problem 9.6



SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1 - \frac{x}{b}\right) \qquad dA = y\,dx = h\left(1 - \frac{x}{b}\right)dx$$

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1 - \frac{x}{b}\right)$$

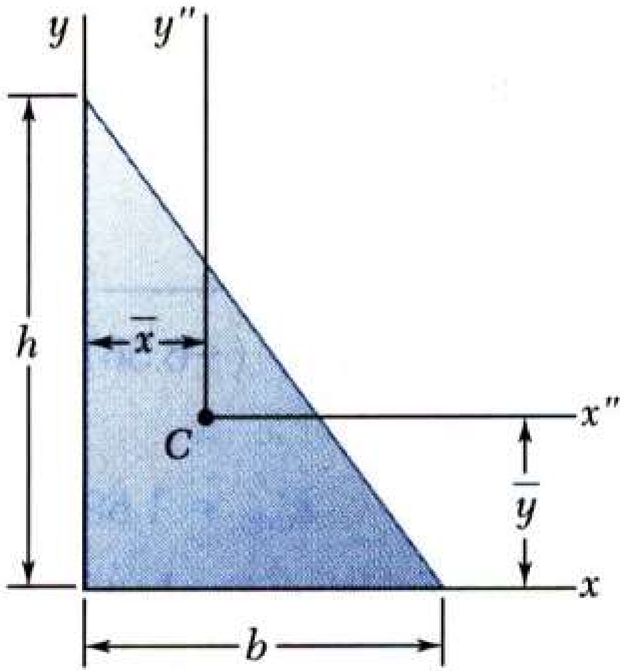
Integrating dI_x from $x = 0$ to $x = b$,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}dA = \int_0^b x\left(\tfrac{1}{2}\right)h^2\left(1 - \frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b\left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b$$

$$\boxed{I_{xy} = \tfrac{1}{24}b^2h^2}$$

Sample Problem 9.6



- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\bar{x} = \tfrac{1}{3}b \qquad \bar{y} = \tfrac{1}{3}h$$

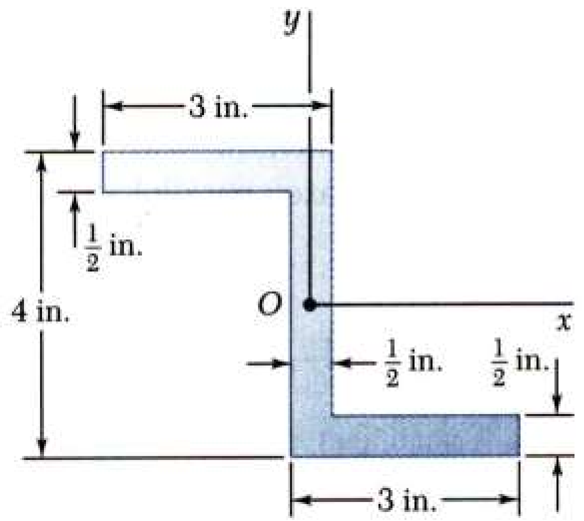
With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

$$\bar{I}_{x''y''} = \tfrac{1}{24}b^2h^2 - \left(\tfrac{1}{3}b\right)\left(\tfrac{1}{3}h\right)\left(\tfrac{1}{2}bh\right)$$

$$\boxed{\bar{I}_{x''y''} = -\tfrac{1}{72}b^2h^2}$$

Sample Problem 9.7



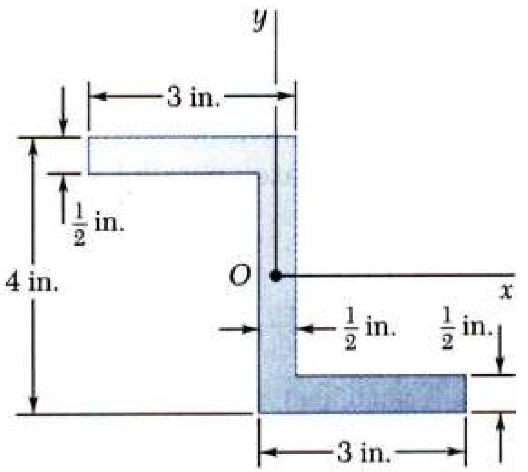
For the section shown, the moments of inertia with respect to the *x* and *y* axes are $I_x = 10.38\ \text{in}^4$ and $I_y = 6.97\ \text{in}^4$.

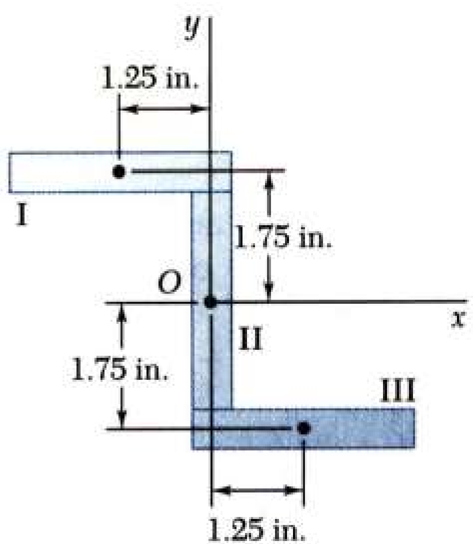
Determine *(a)* the orientation of the principal axes of the section about *O*, and *(b)* the values of the principal moments of inertia about *O*.

SOLUTION:

- Compute the product of inertia with respect to the *xy* axes by dividing the section into three rectangles and applying the parallel axis theorem to each.
- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9. 27).

Sample Problem 9.7





SOLUTION:

- Compute the product of inertia with respect to the *xy* axes by dividing the section into three rectangles.

  Apply the parallel axis theorem to each rectangle,

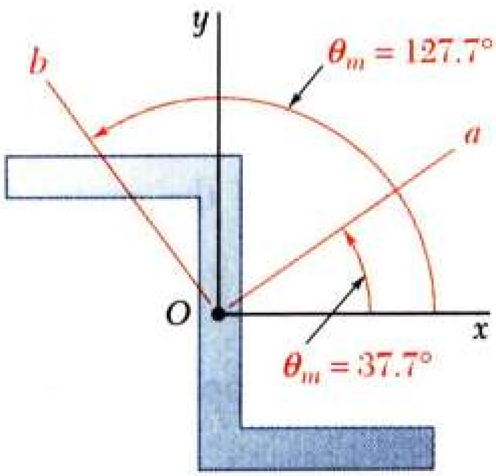
  $$I_{xy} = \sum\left(\bar{I}_{x'y'} + \bar{x}\bar{y}A\right)$$

  Note that the product of inertia with respect to centroidal axes parallel to the *xy* axes is zero for each rectangle.

Rectangle	Area, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A, \text{in}^4$
I	1.5	-1.25	$+1.75$	-3.28
II	1.5	0	0	0
III	1.5	$+1.25$	-1.75	-3.28
				$\sum\bar{x}\bar{y}A = -6.56$

$$\boxed{I_{xy} = \sum\bar{x}\bar{y}A = -6.56\,\text{in}^4}$$

Sample Problem 9.7



$$I_x = 10.38\ \text{in}^4$$

$$I_y = 6.97\ \text{in}^4$$

$$I_{xy} = -6.56\ \text{in}^4$$

- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9. 27).

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

$$2\theta_m = 75.4° \text{ and } 255.4°$$

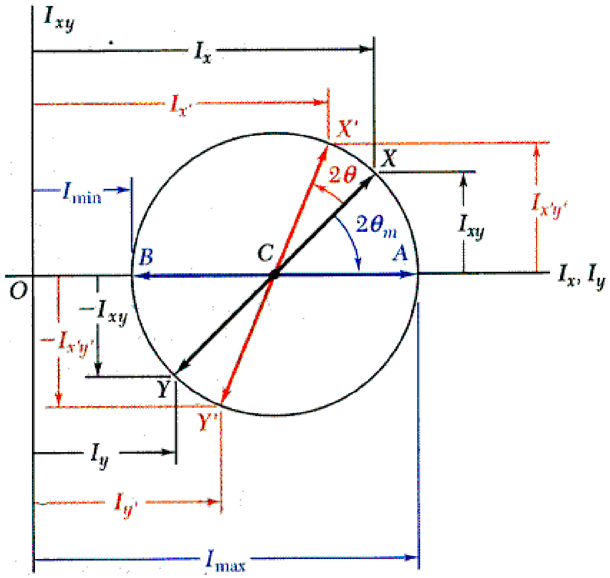
$$\boxed{\theta_m = 37.7° \text{ and } \theta_m = 127.7°}$$

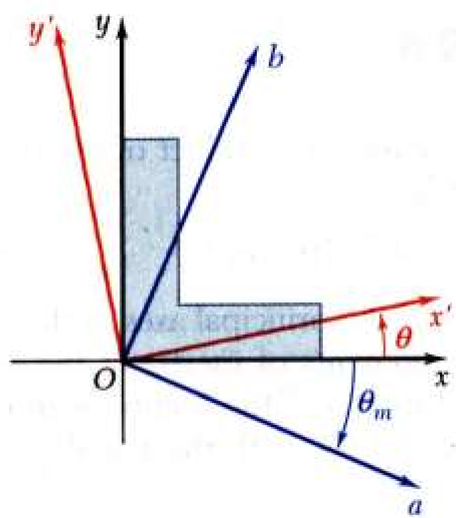
$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2}$$

$$\boxed{\begin{aligned} I_a &= I_{\max} = 15.45\ \text{in}^4 \\ I_b &= I_{\min} = 1.897\ \text{in}^4 \end{aligned}}$$

Mohr's Circle for Moments and Products of Inertia



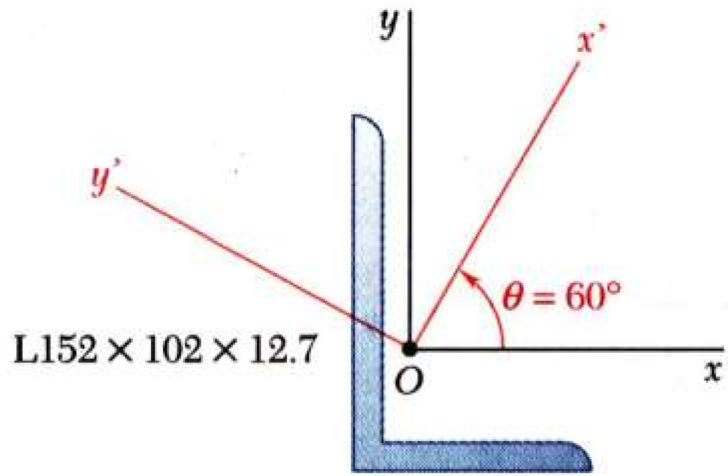


- The moments and product of inertia for an area are plotted as shown and used to construct *Mohr's circle*,

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right) + I_{xy}^2}$$

- Mohr's circle may be used to graphically or analytically determine the moments and product of inertia for any other rectangular axes including the principal axes and principal moments and products of inertia.

Sample Problem 9.8



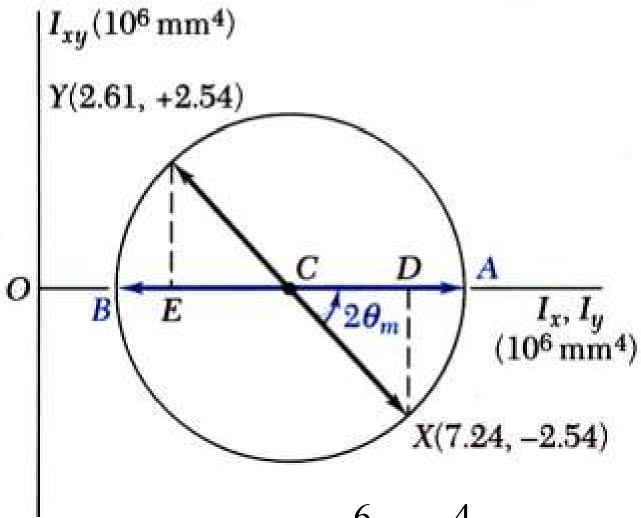
The moments and product of inertia with respect to the *x* and *y* axes are I_x = 7.24x106 mm^4, I_y = 2.61x106 mm^4, and I_{xy} = -2.54x10^6 mm^4.

Using Mohr's circle, determine *(a)* the principal axes about *O*, *(b)* the values of the principal moments about *O*, and *(c)* the values of the moments and product of inertia about the *x'* and *y'* axes

SOLUTION:

- Plot the points (I_x, I_{xy}) and (I_y,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

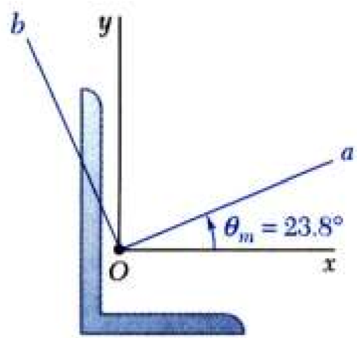
Sample Problem 9.8



$$I_x = 7.24 \times 10^6 \text{ mm}^4$$

$$I_y = 2.61 \times 10^6 \text{ mm}^4$$

$$I_{xy} = -2.54 \times 10^6 \text{ mm}^4$$



SOLUTION:

- Plot the points (I_x, I_{xy}) and (I_y, $-I_{xy}$). Construct Mohr's circle based on the circle diameter between the points.

$$OC = I_{ave} = \tfrac{1}{2}\left(I_x + I_y\right) = 4.925 \times 10^6 \text{ mm}^4$$

$$CD = \tfrac{1}{2}\left(I_x - I_y\right) = 2.315 \times 10^6 \text{ mm}^4$$

$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \text{ mm}^4$$

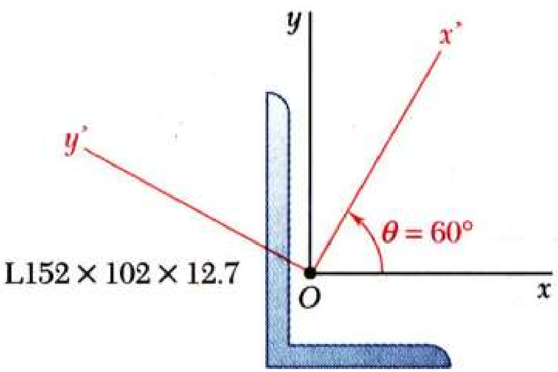
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

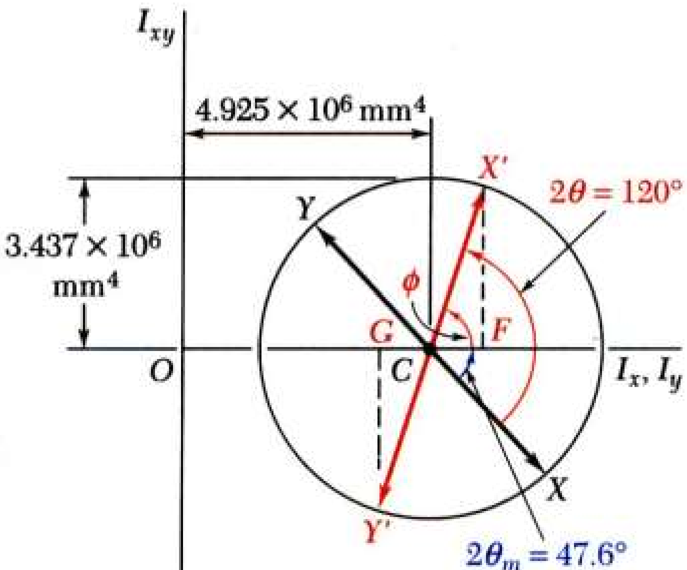
$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6° \qquad \boxed{\theta_m = 23.8°}$$

$$I_{max} = OA = I_{ave} + R \qquad \boxed{I_{max} = 8.36 \times 10^6 \text{ mm}^4}$$

$$I_{min} = OB = I_{ave} - R \qquad \boxed{I_{min} = 1.49 \times 10^6 \text{ mm}^4}$$

Sample Problem 9.8





$$OC = I_{ave} = 4.925 \times 10^6 \text{mm}^4$$

$$R = 3.437 \times 10^6 \text{mm}^4$$

- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

The points *X'* and *Y'* corresponding to the *x'* and *y'* axes are obtained by rotating *CX* and *CY* counterclockwise through an angle $\Theta = 2(60^o) = 120^o$. The angle that *CX'* forms with the *x'* axes is $\phi = 120^o - 47.6^o = 72.4^o$.

$$I_{x'} = OF = OC + CX' \cos\varphi = I_{ave} + R\cos 72.4^o$$

$$\boxed{I_{x'} = 5.96 \times 10^6 \text{mm}^4}$$

$$I_{y'} = OG = OC - CY' \cos\varphi = I_{ave} - R\cos 72.4^o$$

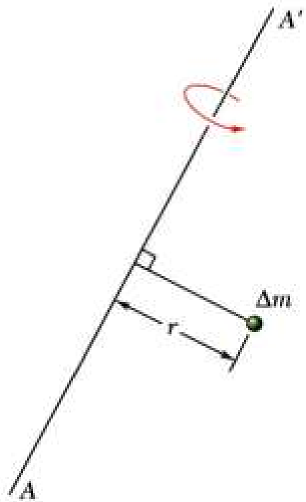
$$\boxed{I_{y'} = 3.89 \times 10^6 \text{mm}^4}$$

$$I_{x'y'} = FX' = CY' \sin\varphi = R\sin 72.4^o$$

$$\boxed{I_{x'y'} = 3.28 \times 10^6 \text{mm}^4}$$

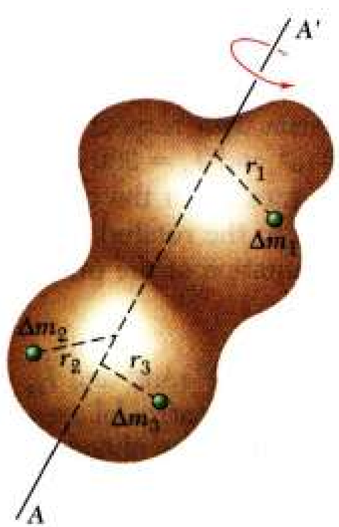
Moment of Inertia of a Mass

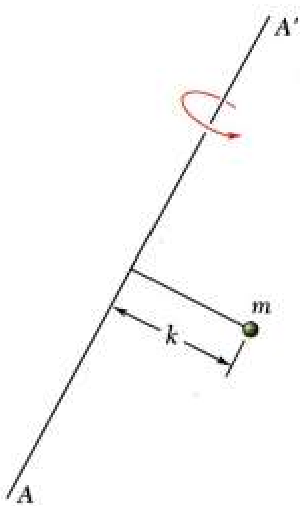
Moment of Inertia of a Mass



- Angular acceleration about the axis *AA'* of the small mass Δm due to the application of a couple is proportional to $r^2\Delta m$.

  $r^2\Delta m$ = *moment of inertia* of the mass Δm with respect to the axis *AA'*





- For a body of mass *m* the resistance to rotation about the axis *AA' is*

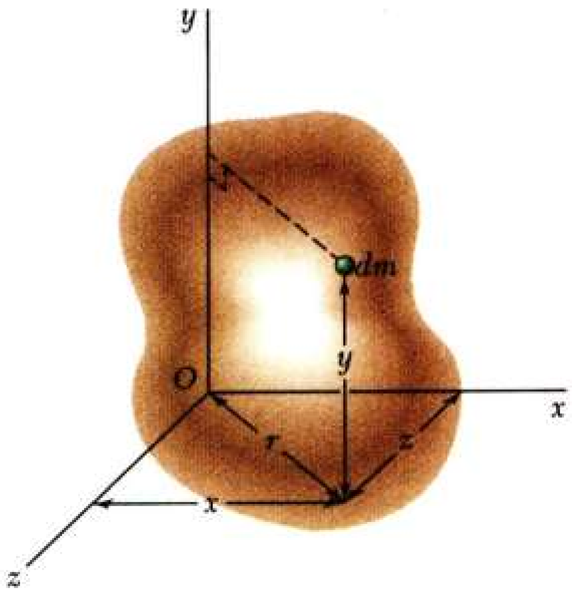
$$I = r_1^2\Delta m + r_2^2\Delta m + r_3^2\Delta m + \cdots$$

$$= \int r^2 dm = \textit{mass moment of inertia}$$

- The radius of gyration for a concentrated mass with equivalent mass moment of inertia is

$$I = k^2 m \qquad k = \sqrt{\frac{I}{m}}$$

Moment of Inertia of a Mass



- Moment of inertia with respect to the y coordinate axis is

$$I_y = \int r^2 dm = \int \left(z^2 + x^2\right) dm$$

- Similarly, for the moment of inertia with respect to the x and z axes,

$$I_x = \int \left(y^2 + z^2\right) dm$$

$$I_z = \int \left(x^2 + y^2\right) dm$$

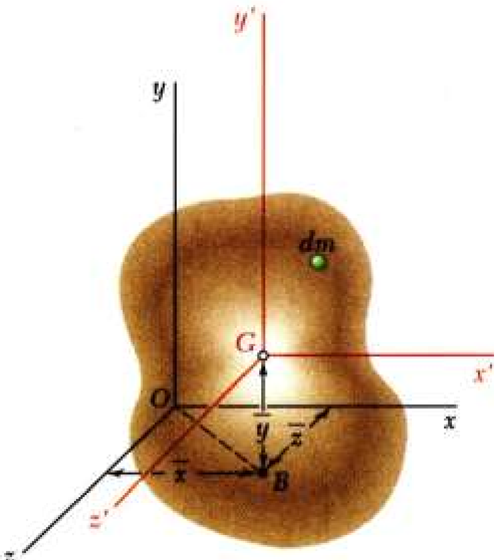
- In SI units,

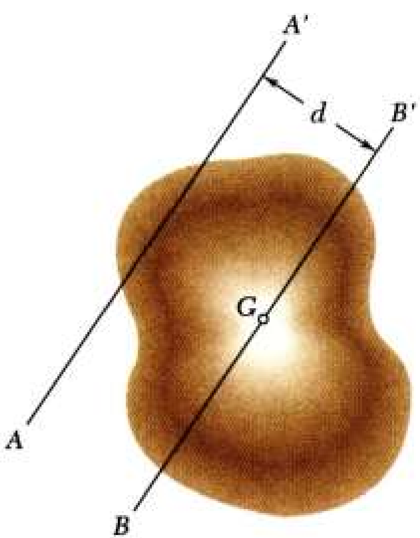
$$I = \int r^2 dm = \left(\mathrm{kg} \cdot \mathrm{m}^2\right)$$

In U.S. customary units,

$$I = \left(slug \cdot ft^2\right) = \left(\frac{lb \cdot s^2}{ft} ft^2\right) = \left(lb \cdot ft \cdot s^2\right)$$

Parallel Axis Theorem





- For the rectangular axes with origin at O and parallel centroidal axes,

$$I_x = \int \left(y^2 + z^2\right) dm = \int \left[\left(y' + \bar{y}\right)^2 + \left(z' + \bar{z}\right)^2\right] dm$$

$$= \int \left(y'^2 + z'^2\right) dm + 2\bar{y}\int y' dm + 2\bar{z}\int z' dm + \left(\bar{y}^2 + \bar{z}^2\right)\int dm$$

$$I_x = \bar{I}_{x'} + m\left(\bar{y}^2 + \bar{z}^2\right)$$

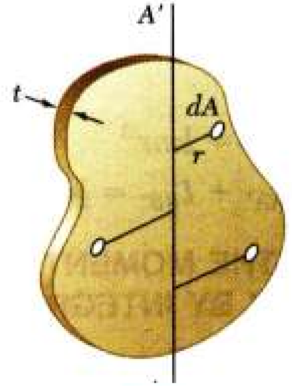
$$I_y = \bar{I}_{y'} + m\left(\bar{z}^2 + \bar{x}^2\right)$$

$$I_z = \bar{I}_{z'} + m\left(\bar{x}^2 + \bar{y}^2\right)$$

- Generalizing for any axis AA' and a parallel centroidal axis,

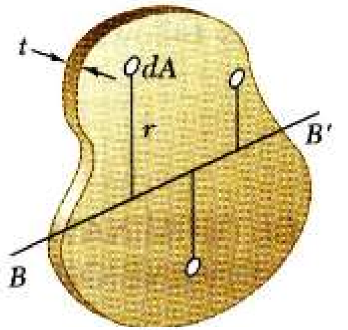
$$I = \bar{I} + md^2$$

Moments of Inertia of Thin Plates



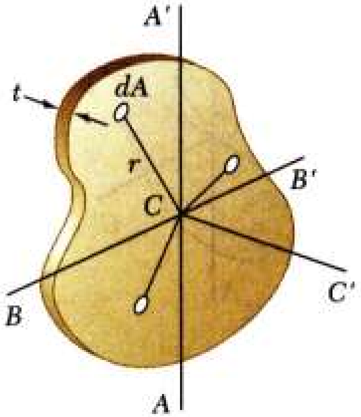
- For a thin plate of uniform thickness t and homogeneous material of density ρ, the mass moment of inertia with respect to axis AA' contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t \, I_{AA',area}$$



- Similarly, for perpendicular axis BB' which is also contained in the plate,

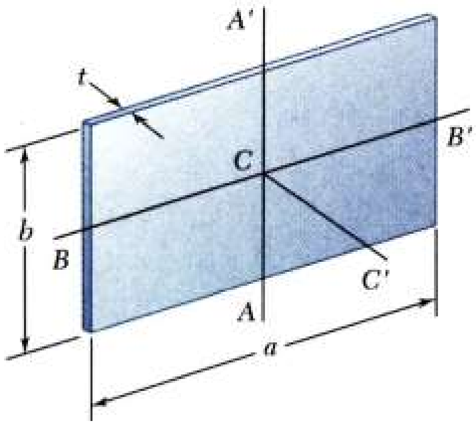
$$I_{BB'} = \rho t \, I_{BB',area}$$



- For the axis CC' which is perpendicular to the plate,

$$I_{CC'} = \rho t \, J_{C,area} = \rho t \left(I_{AA',area} + I_{BB',area} \right)$$
$$= I_{AA'} + I_{BB'}$$

Moments of Inertia of Thin Plates

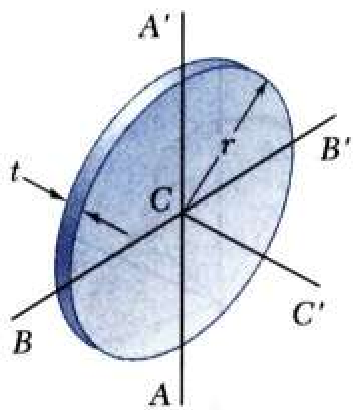


- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t\, I_{AA',area} = \rho t\left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$

$$I_{BB'} = \rho t\, I_{BB',area} = \rho t\left(\tfrac{1}{12} a b^3\right) = \tfrac{1}{12} m b^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12} m\left(a^2 + b^2\right)$$

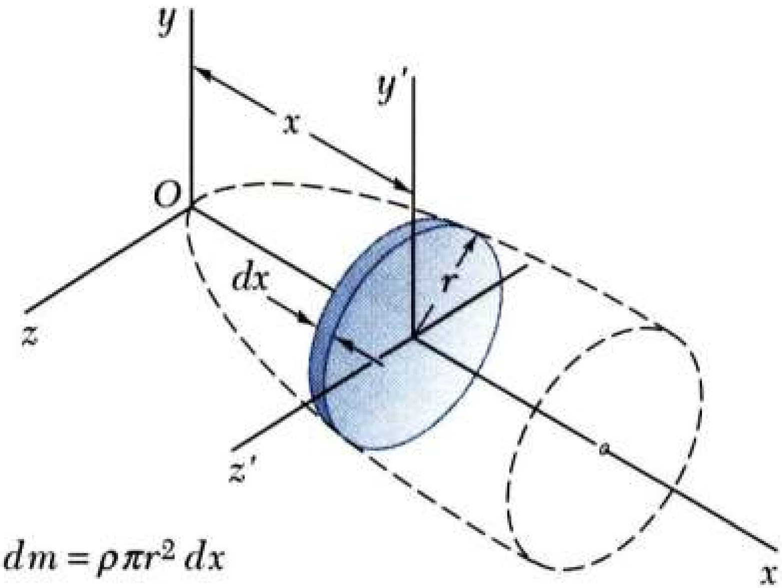


- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t\, I_{AA',area} = \rho t\left(\tfrac{1}{4} \pi r^4\right) = \tfrac{1}{4} m r^2$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2} m r^2$$

Moments of Inertia of a 3D Body by Integration



$$dm = \rho \pi r^2 \, dx$$

$$dI_x = \frac{1}{2} r^2 \, dm$$

$$dI_y = dI_{y'} + x^2 \, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2 \, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

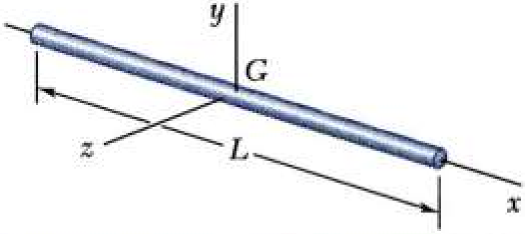
- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

$$I = \rho \int r^2 dV$$

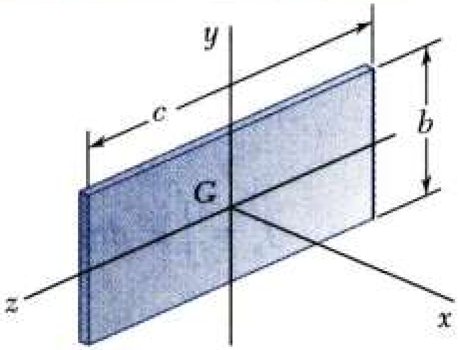
- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for *dm*.

- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.

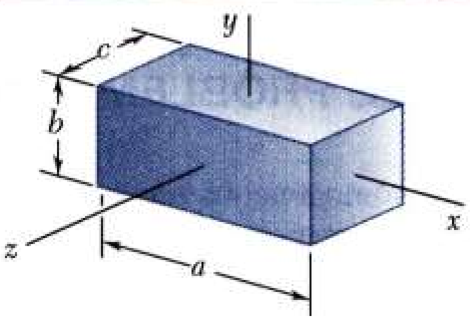
Moments of Inertia of Common Geometric Shapes



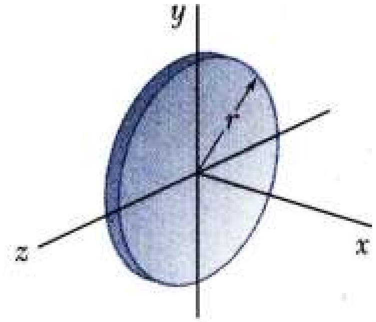
$$I_y = I_z = \frac{1}{12} mL^2$$



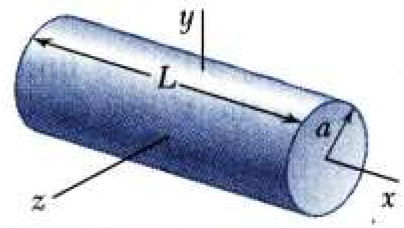
$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} mc^2$$
$$I_z = \frac{1}{12} mb^2$$



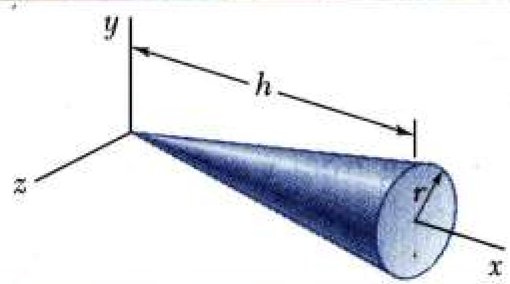
$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} m(c^2 + a^2)$$
$$I_z = \frac{1}{12} m(a^2 + b^2)$$



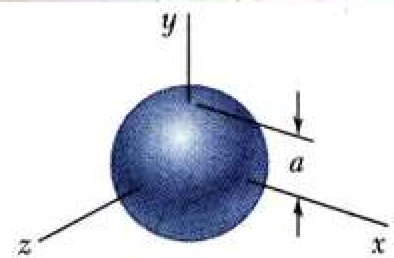
$$I_x = \frac{1}{2} mr^2$$
$$I_y = I_z = \frac{1}{4} mr^2$$



$$I_x = \frac{1}{2} ma^2$$
$$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$$

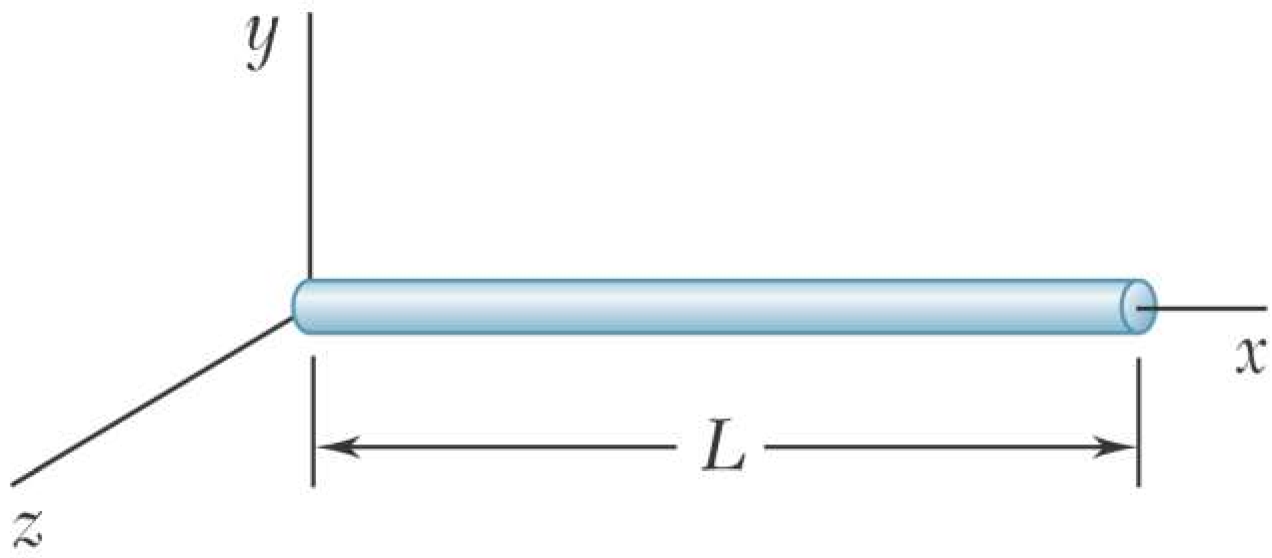


$$I_x = \frac{3}{10} ma^2$$
$$I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$$



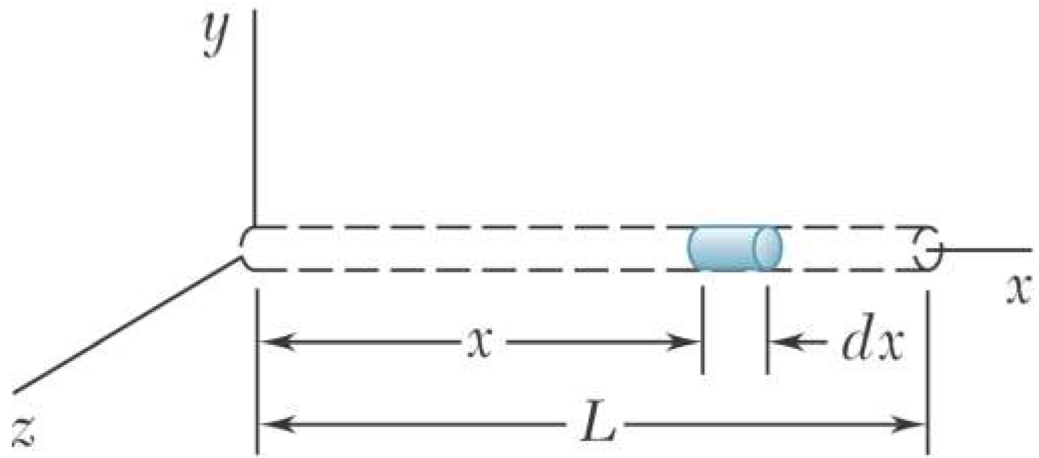
$$I_x = I_y = I_z = \frac{2}{5} ma^2$$

Sample Problem 9.9



Determine the moments of inertia of a slender rod of length L and mass m with respect to an axis which is perpendicular to the rod and passes through one end of the rod

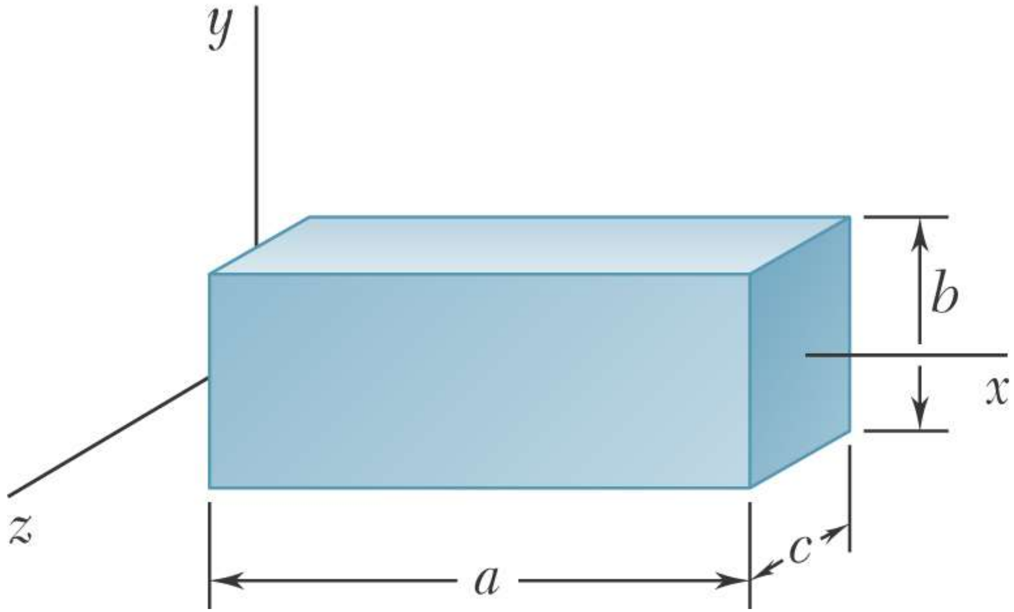
Sample Problem 9.9



$$dm = \frac{\mathrm{m}}{\mathrm{L}}dx$$

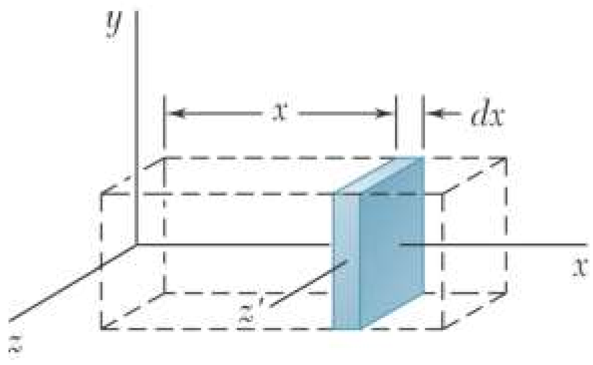
$$I_y = \int x^2 dm = \int_0^L x^2 \frac{\mathrm{m}}{\mathrm{L}}dx = \frac{1}{3}mL^2$$

Sample Problem 9.10



For the homogeneous rectangular prism shown, determine the moments of inertia with respect to the z axis

Sample Problem 9.10



$$dm = \rho bcdx$$

$$dI_{z'} = \frac{1}{12}b^2 dm$$

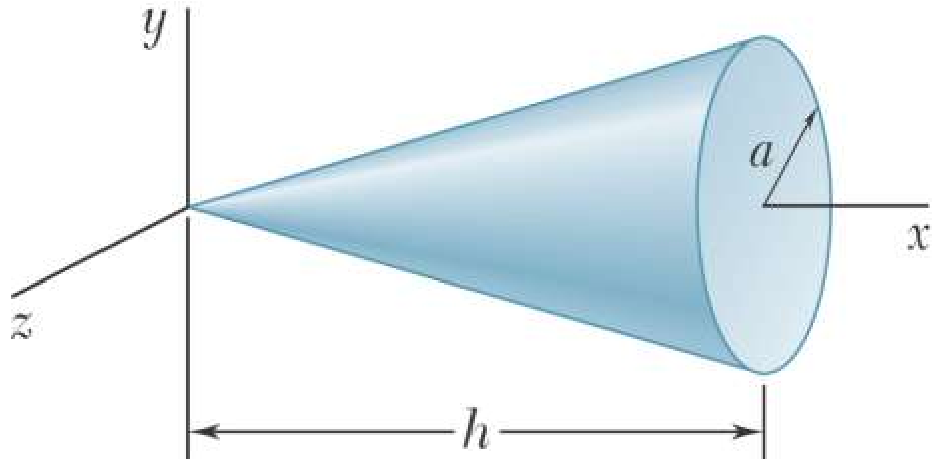
$$dI_z = dI_{z'} + x^2 dm = \left(\frac{1}{12}b^2 + x^2\right)\rho bcdx$$

$$I_z = \int_0^a \left(\frac{1}{12}b^2 + x^2\right)\rho bcdx$$

$$= \left(\frac{1}{12}b^2 + \frac{1}{3}a^2\right)\rho abc$$

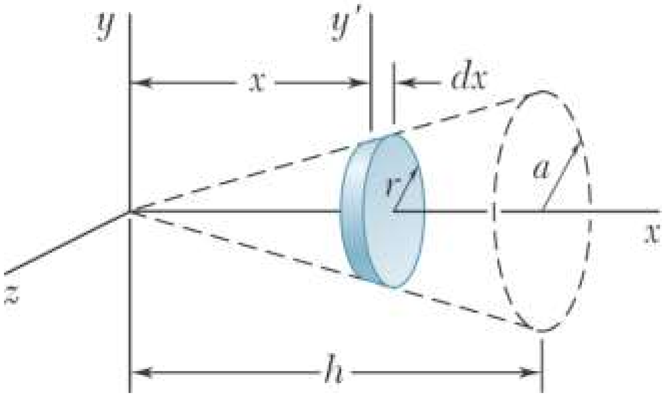
$$= m\left(\frac{1}{12}b^2 + \frac{1}{3}a^2\right) = \frac{1}{12}m\left(4a^2 + b^2\right)$$

Sample Problem 9.11



Determine the moments of inertia of a right circular cone with respect to (a) its longitudinal axis, (b) an axis through the apex of the cone and perpendicular to its longitudinal axis, (c) an axis through the centroid of the cone and perpendicular to its longitudinal axis

Sample Problem 9.11



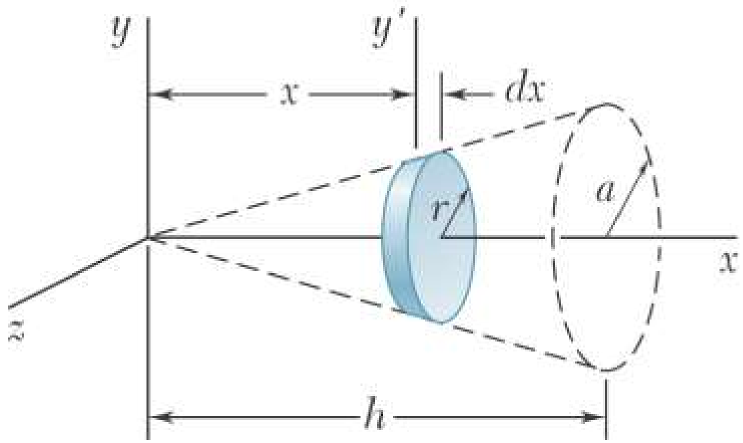
$$dm = \rho\pi r^2 dx = \rho\pi\left(\frac{a}{h}x\right)^2 dx$$

$$dI_x = \frac{1}{2}r^2 dm = \frac{1}{2}\left(\frac{a}{h}x\right)^2 \rho\pi\left(\frac{a}{h}x\right)^2 dx$$

$$I_x = \int_0^h \frac{1}{2}\left(\frac{a}{h}x\right)^2 \rho\pi\left(\frac{a}{h}x\right)^2 dx = \frac{1}{2}\rho\pi\frac{a^4}{h^4}\frac{h^5}{5} = \frac{1}{10}\rho\pi a^4 h$$

$$= \frac{3}{10}a^2\left(\frac{1}{3}\rho\pi a^2 h\right) = \frac{3}{10}ma^2$$

Sample Problem 9.11



$$dm = \rho\pi r^2 dx = \rho\pi\left(\frac{a}{h}x\right)^2 dx$$

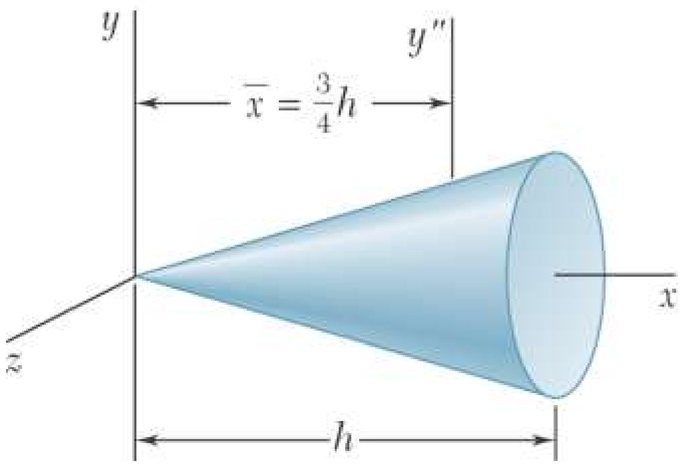
$$dI_y = dI_{y'} + x^2 dm = \frac{1}{4}r^2 dm + x^2 dm$$

$$= \left(\frac{1}{4}r^2 + x^2\right)dm$$

$$= \left(\frac{1}{4}\left(\frac{a}{h}x\right)^2 + x^2\right)\rho\pi\left(\frac{a}{h}x\right)^2 dx$$

$$I_y = \int_0^h \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2}+1\right)x^4 dx = \frac{3}{5}\left(\frac{1}{4}a^2 + h^2\right)\frac{1}{3}\rho\pi a^2 h = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$$

Sample Problem 9.11



$$I_y = \bar{I}_{y''} + m\bar{x}^2$$

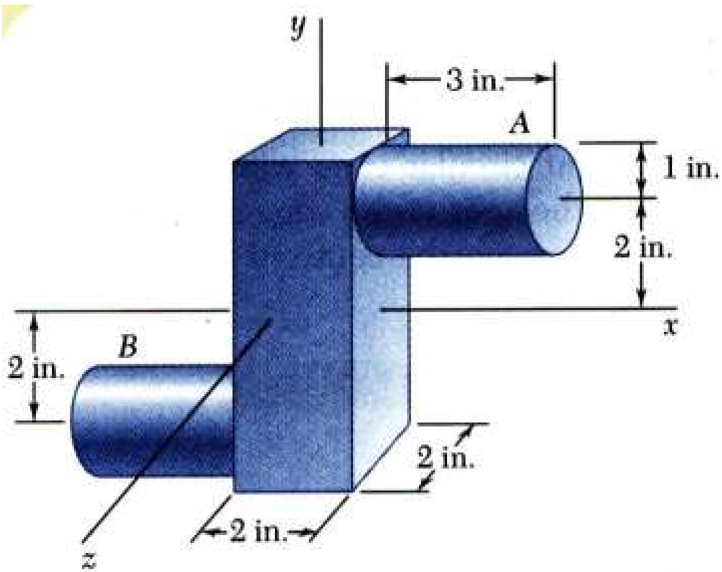
$$\bar{x} = \frac{3}{4}h$$

$$\bar{I}_{y''} = I_y - m\bar{x}^2$$

$$= \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right) - m\left(\frac{3}{4}h\right)^2$$

$$= \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$$

Sample Problem 9.12



Determine the moments of inertia of the steel forging with respect to the *xyz* coordinate axes, knowing that the specific weight of steel is 490 lb/ft3.

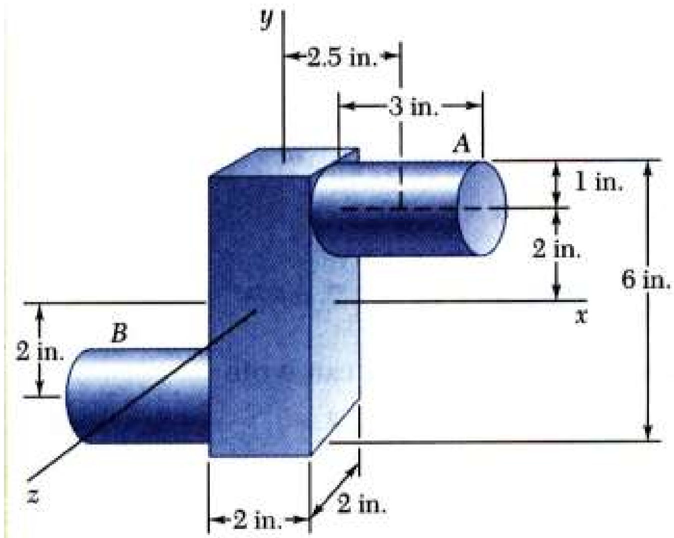
<u>SOLUTION</u>:

- With the forging divided into a prism and two cylinders, compute the mass and moments of inertia of each component with respect to the *xyz* axes using the parallel axis theorem.
- Add the moments of inertia from the components to determine the total moments of inertia for the forging.

Sample Problem 9.12

SOLUTION:

- Compute the moments of inertia of each component with respect to the *xyz* axes.



each cylinder :

$$m = \frac{\gamma V}{g} = \frac{(490\,\mathrm{lb/ft^3})(\pi \times 1^2 \times 3)\mathrm{in}^3}{(1728\,\mathrm{in}^3/\mathrm{ft}^3)(32.2\,\mathrm{ft/s^2})}$$

$$m = 0.0829\ \mathrm{lb \cdot s^2/ft}$$

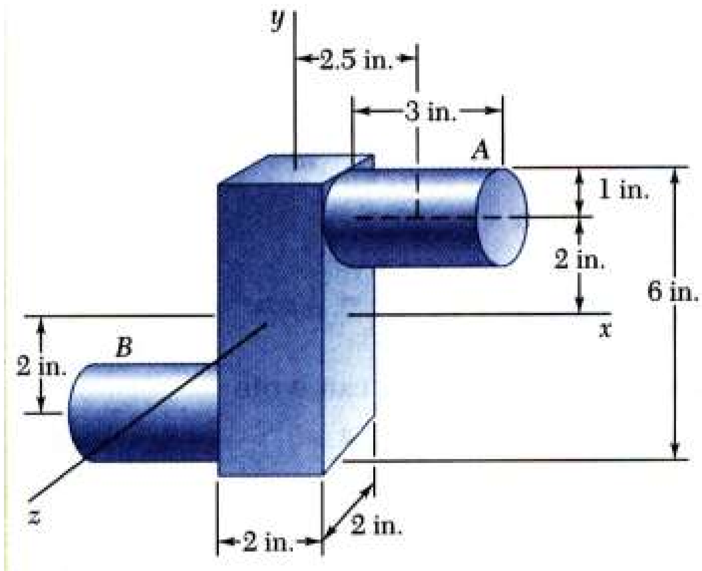
cylinders $(a = 1\mathrm{in.}, L = 3\mathrm{in.}, \bar{x} = 2.5\mathrm{in.}, \bar{y} = 2\mathrm{in.})$:

$$I_x = \tfrac{1}{2} m a^2 + m\bar{y}^2$$
$$= \tfrac{1}{2}(0.0829)\left(\tfrac{1}{12}\right)^2 + (0.0829)\left(\tfrac{2}{12}\right)^2$$
$$= 2.59 \times 10^{-3}\,\mathrm{lb \cdot ft \cdot s^2}$$

$$I_y = \tfrac{1}{12} m\left[3a^2 + L^2\right] + m\bar{x}^2$$
$$= \tfrac{1}{12}(0.0829)\left[3\left(\tfrac{1}{12}\right)^2 + \left(\tfrac{3}{12}\right)^2\right] + (0.0829)\left(\tfrac{2.5}{12}\right)^2$$
$$= 4.17 \times 10^{-3}\,\mathrm{lb \cdot ft \cdot s^2}$$

$$I_y = \tfrac{1}{12} m\left[3a^2 + L^2\right] + m\left[\bar{x}^2 + \bar{y}^2\right]$$
$$= \tfrac{1}{12}(0.0829)\left[3\left(\tfrac{1}{12}\right)^2 + \left(\tfrac{3}{12}\right)^2\right] + (0.0829)\left[\left(\tfrac{2.5}{12}\right)^2 + \left(\tfrac{2}{12}\right)^2\right]$$
$$= 6.48 \times 10^{-3}\,\mathrm{lb \cdot ft \cdot s^2}$$

Sample Problem 9.12



prism:

$$m = \frac{\gamma V}{g} = \frac{(490\,\text{lb/ft}^3)(2\times 2\times 6)\text{in}^3}{(1728\,\text{in}^3/\text{ft}^3)(32.2\,\text{ft/s}^2)}$$

$$m = 0.211\ \text{lb}\cdot\text{s}^2/\text{ft}$$

prism ($a = 2$ in., $b = 6$ in., $c = 2$ in.):

$$I_x = I_z = \tfrac{1}{12}m\left[b^2 + c^2\right] = \tfrac{1}{12}(0.211)\left[\left(\tfrac{6}{12}\right)^2 + \left(\tfrac{2}{12}\right)^2\right]$$

$$= 4.88\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = \tfrac{1}{12}m\left[c^2 + a^2\right] = \tfrac{1}{12}(0.211)\left[\left(\tfrac{2}{12}\right)^2 + \left(\tfrac{2}{12}\right)^2\right]$$

$$= 0.977\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

- Add the moments of inertia from the components to determine the total moments of inertia.

$$I_x = 4.88\times 10^{-3} + 2\left(2.59\times 10^{-3}\right)$$

$$\boxed{I_x = 10.06\times 10^{-3}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

$$I_y = 0.977\times 10^{-3} + 2\left(4.17\times 10^{-3}\right)$$

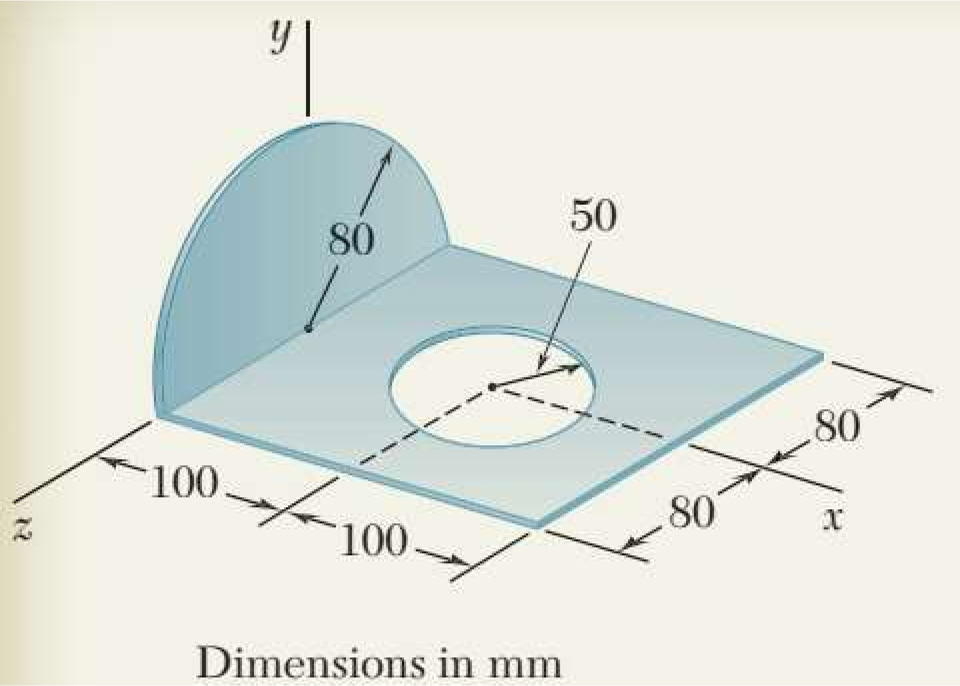
$$\boxed{I_y = 9.32\times 10^{-3}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

$$I_z = 4.88\times 10^{-3} + 2\left(6.48\times 10^{-3}\right)$$

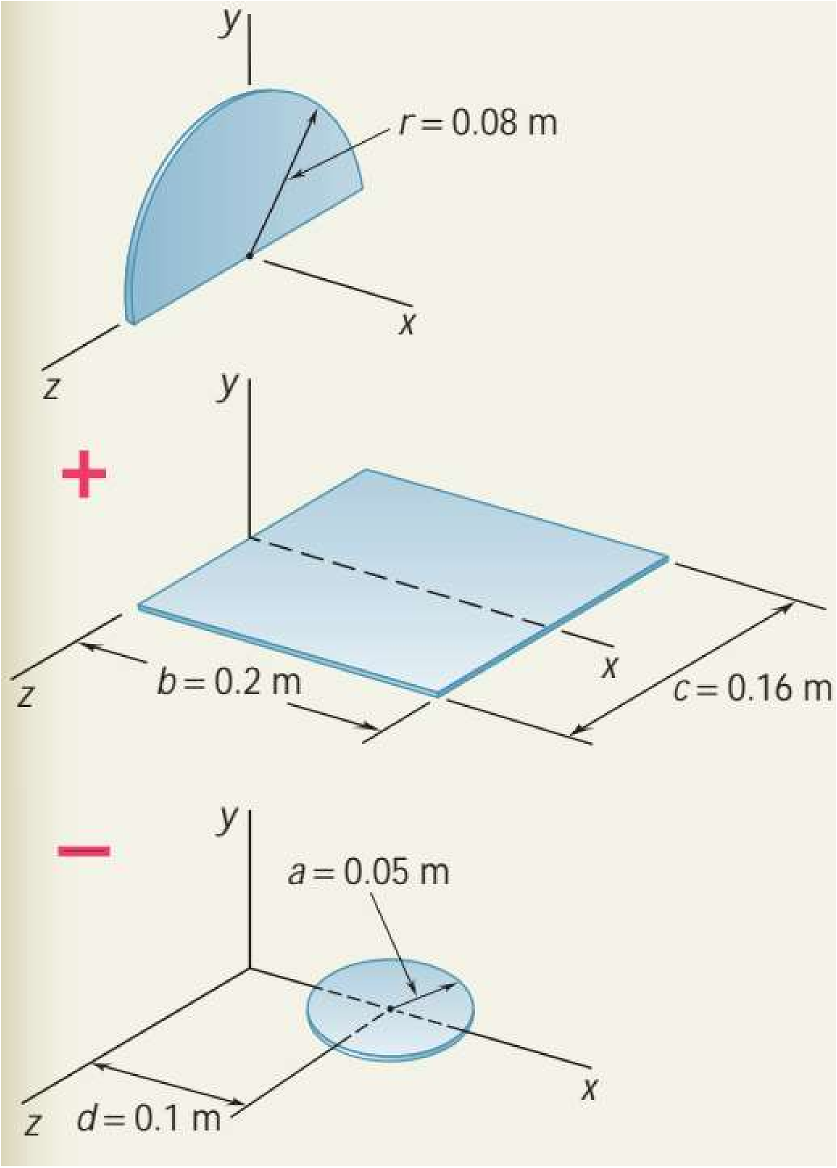
$$\boxed{I_z = 17.84\times 10^{-3}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

Sample Problem 9.13

A thin steel plate which is 4 mm thick is cut and bent to form the machine part shown. Knowing that the density of steel is 7850 kg/m^3, determine the moments of inertia of the machine part with respect to the coordinate axes.



Sample Problem 9.13



Sample Problem 9.13

We observe that the machine part consists of a semicircular plate and a rectangular plate from which a circular plate has been removed.

Computation of Masses. ***Semicircular Plate***

$$V_1 = \tfrac{1}{2}\pi r^2 t = \tfrac{1}{2}\pi(0.08\text{ m})^2(0.004\text{ m}) = 40.21 \times 10^{-6}\text{ m}^3$$

$$m_1 = \rho V_1 = (7.85 \times 10^3\text{ kg/m}^3)(40.21 \times 10^{-6}\text{ m}^3) = 0.3156\text{ kg}$$

Rectangular Plate

$$V_2 = (0.200\text{ m})(0.160\text{ m})(0.004\text{ m}) = 128 \times 10^{-6}\text{ m}^3$$

$$m_2 = \rho V_2 = (7.85 \times 10^3\text{ kg/m}^3)(128 \times 10^{-6}\text{ m}^3) = 1.005\text{ kg}$$

Circular Plate

$$V_3 = \pi a^2 t = \pi(0.050\text{ m})^2(0.004\text{ m}) = 31.42 \times 10^{-6}\text{ m}^3$$

$$m_3 = \rho V_3 = (7.85 \times 10^3\text{ kg/m}^3)(31.42 \times 10^{-6}\text{ m}^3) = 0.2466\text{ kg}$$

Sample Problem 9.13

Moments of Inertia. Using the method presented in Sec. 9.13, we compute the moments of inertia of each component.

Semicircular Plate. From Fig. 9.28, we observe that for a circular plate of mass m and radius r

$$I_x = \tfrac{1}{2}mr^2 \qquad I_y = I_z = \tfrac{1}{4}mr^2$$

Because of symmetry, we note that for a semicircular plate

$$I_x = \tfrac{1}{2}(\tfrac{1}{2}mr^2) \qquad I_y = I_z = \tfrac{1}{2}(\tfrac{1}{4}mr^2)$$

Since the mass of the semicircular plate is $m_1 = \tfrac{1}{2}m$, we have

$$I_x = \tfrac{1}{2}m_1r^2 = \tfrac{1}{2}(0.3156\text{ kg})(0.08\text{ m})^2 = 1.010 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$
$$I_y = I_z = \tfrac{1}{4}(\tfrac{1}{2}mr^2) = \tfrac{1}{4}m_1r^2 = \tfrac{1}{4}(0.3156\text{ kg})(0.08\text{ m})^2 = 0.505 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$

Sample Problem 9.13

Rectangular Plate

$$I_x = \tfrac{1}{12}m_2c^2 = \tfrac{1}{12}(1.005\text{ kg})(0.16\text{ m})^2 = 2.144 \times 10^{-3}\text{kg} \cdot \text{m}^2$$
$$I_z = \tfrac{1}{3}m_2b^2 = \tfrac{1}{3}(1.005\text{kg})(0.2\text{m})^2 = 13.400 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$
$$I_y = I_x + I_z = (2.144 + 13.400)(10^{-3}) = 15.544 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$

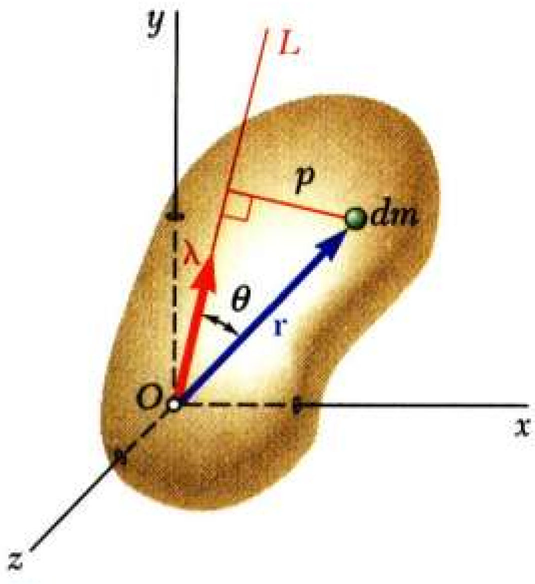
Circular Plate

$$I_x = \tfrac{1}{4}m_3a^2 = \tfrac{1}{4}(0.2466\text{ kg})(0.05\text{m})^2 = 0.154 \times 10^{-3}\text{kg} \cdot \text{m}^2$$
$$I_y = \tfrac{1}{2}m_3a^2 + m_3d^2$$
$$= \tfrac{1}{2}(0.2466\text{ kg})(0.05\text{ m})^2 + (0.2466\text{ kg})(0.1\text{ m})^2 = 2.774 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$
$$I_z = \tfrac{1}{4}m_3a^2 + m_3d^2 = \tfrac{1}{4}(0.2466\text{ kg})(0.05\text{ m})^2 + (0.2466\text{ kg})(0.1\text{ m})^2$$
$$= 2.620 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$

Entire Machine Part

$$I_x = (1.010 + 2.144 - 0.154)(10^{-3})\text{ kg} \cdot \text{m}^2 \qquad I_x = 3.00 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$
$$I_y = (0.505 + 15.544 - 2.774)(10^{-3})\text{ kg} \cdot \text{m}^2 \qquad I_y = 13.28 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$
$$I_z = (0.505 + 13.400 - 2.620)(10^{-3})\text{ kg} \cdot \text{m}^2 \qquad I_z = 11.29 \times 10^{-3}\text{ kg} \cdot \text{m}^2$$

Moment of Inertia With Respect to an Arbitrary Axis



- I_{OL} = moment of inertia with respect to axis OL

$$I_{OL} = \int p^2 dm = \int \left|\vec{\lambda} \times \vec{r}\right|^2 dm$$

- Expressing $\vec{\lambda}$ and $\vec{r}$ in terms of the vector components and expanding yields

$$I_{OL} = I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

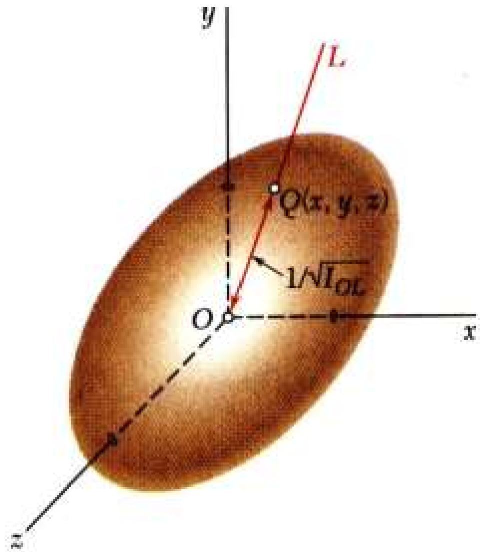
- The definition of the mass products of inertia of a mass is an extension of the definition of product of inertia of an area

$$I_{xy} = \int xy\, dm = \bar{I}_{x'y'} + m\bar{x}\bar{y}$$

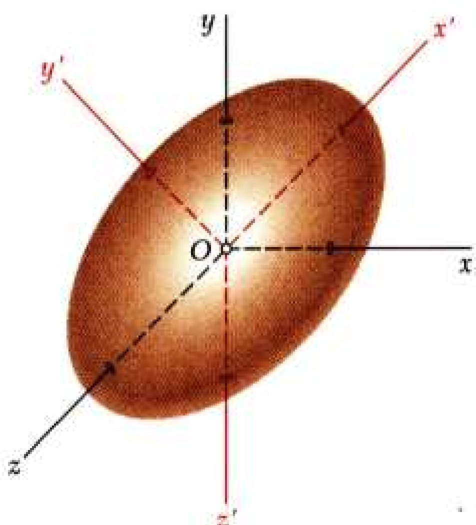
$$I_{yz} = \int yz\, dm = \bar{I}_{y'z'} + m\bar{y}\bar{z}$$

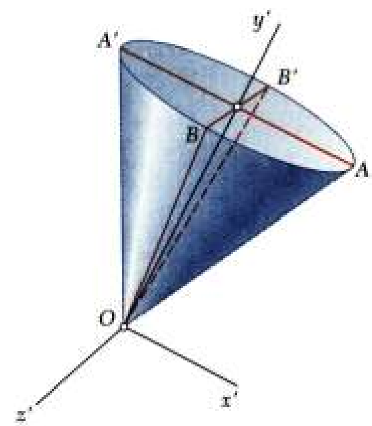
$$I_{zx} = \int zx\, dm = \bar{I}_{z'x'} + m\bar{z}\bar{x}$$

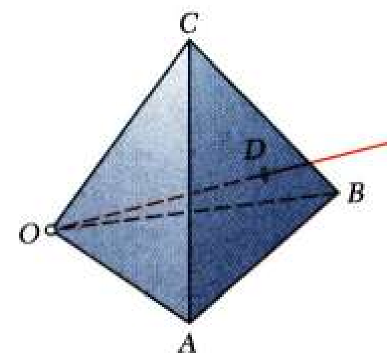
Ellipsoid of Inertia. Principal Axes of Inertia of a Mass



- Assume the moment of inertia of a body has been computed for a large number of axes *OL* and that point *Q* is plotted on each axis at a distance $OQ = 1/\sqrt{I_{OL}}$
- The locus of points *Q* forms a surface known as the *ellipsoid of inertia* which defines the moment of inertia of the body for any axis through *O*.
- *x'*,*y'*,*z'* axes may be chosen which are the *principal axes of inertia* for which the products of inertia are zero and the moments of inertia are the *principal moments of inertia*.

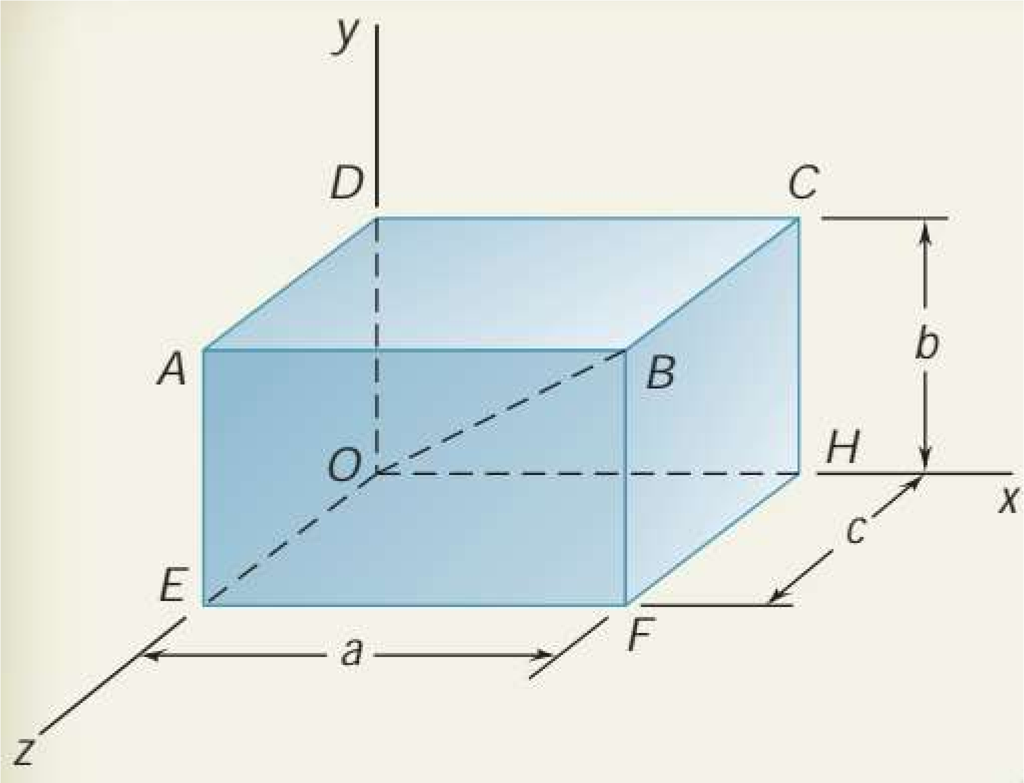




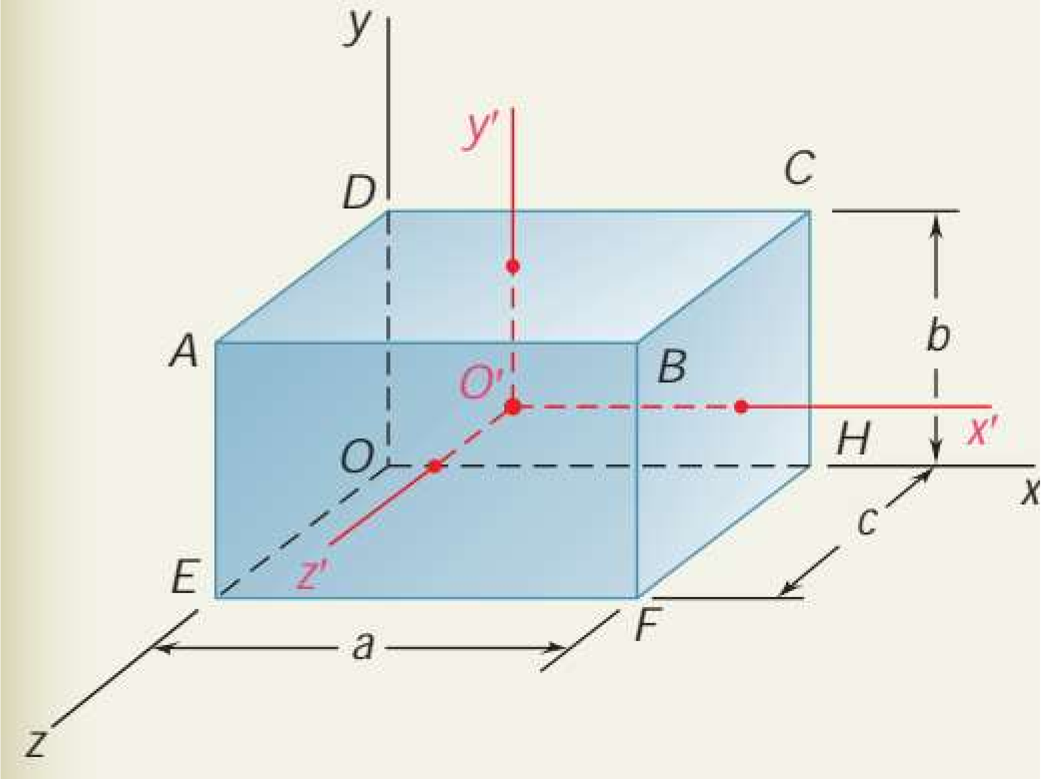


Sample Problem 9.14

Consider a rectangular prism of mass m and sides a, b, c. Determine (*a*) the moments and products of inertia of the prism with respect to the coordinate axes shown, (*b*) its moment of inertia with respect to the diagonal OB.



Sample Problem 9.14



Sample Problem 9.14

a. Moments and Products of Inertia with Respect to the Coordinate Axes. ***Moments of Inertia.*** Introducing the centroidal axes x', y', z', with respect to which the moments of inertia are given in Fig. 9.28, we apply the parallel-axis theorem:

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2) = \tfrac{1}{12}m(b^2 + c^2) + m(\tfrac{1}{4}b^2 + \tfrac{1}{4}c^2)$$

$$I_x = \tfrac{1}{3}m(b^2 + c^2) \quad \blacktriangleleft$$

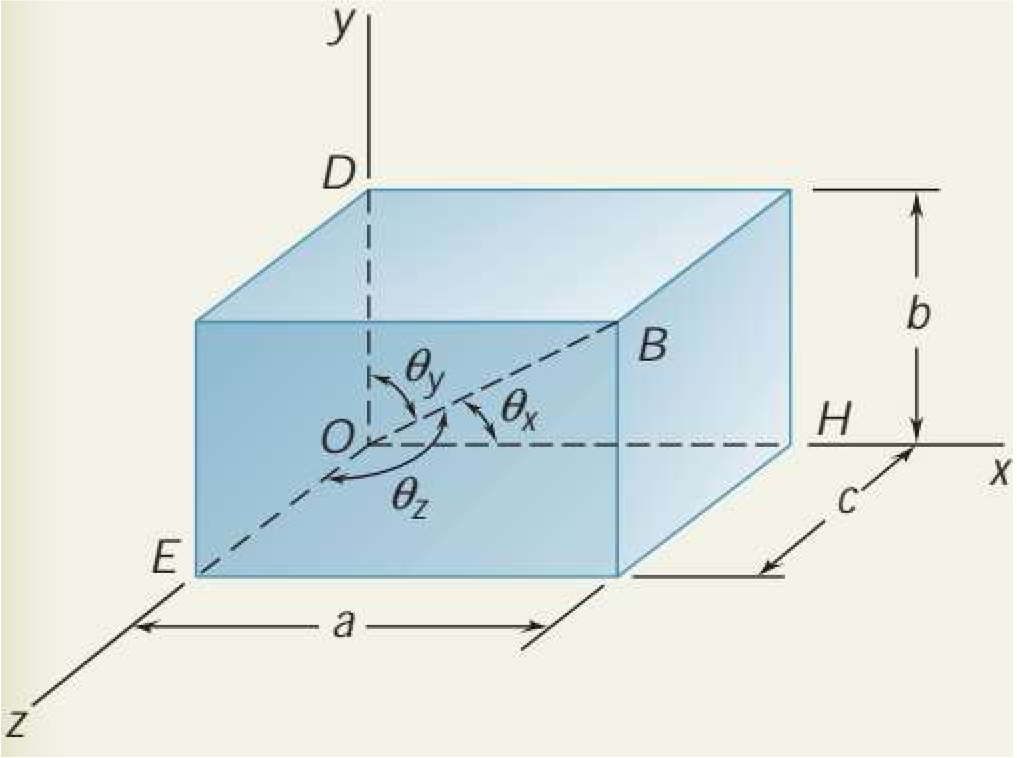
Similarly, $I_y = \tfrac{1}{3}m(c^2 + a^2) \qquad I_z = \tfrac{1}{3}m(a^2 + b^2) \quad \blacktriangleleft$

Products of Inertia. Because of symmetry, the products of inertia with respect to the centroidal axes x', y', z' are zero, and these axes are principal axes of inertia. Using the parallel-axis theorem, we have

$$I_{xy} = \bar{I}_{x'y'} + m\bar{x}\bar{y} = 0 + m(\tfrac{1}{2}a)(\tfrac{1}{2}b) \qquad I_{xy} = \tfrac{1}{4}mab \quad \blacktriangleleft$$

Similarly, $I_{yz} = \tfrac{1}{4}mbc \qquad I_{zx} = \tfrac{1}{4}mca \quad \blacktriangleleft$

Sample Problem 9.14



Sample Problem 9.14

b. Moment of Inertia with Respect to *OB*. We recall Eq. (9.46):

$$I_{OB} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

where the direction cosines of OB are

$$\lambda_x = \cos\theta_x = \frac{OH}{OB} = \frac{a}{(a^2 + b^2 + c^2)^{1/2}}$$

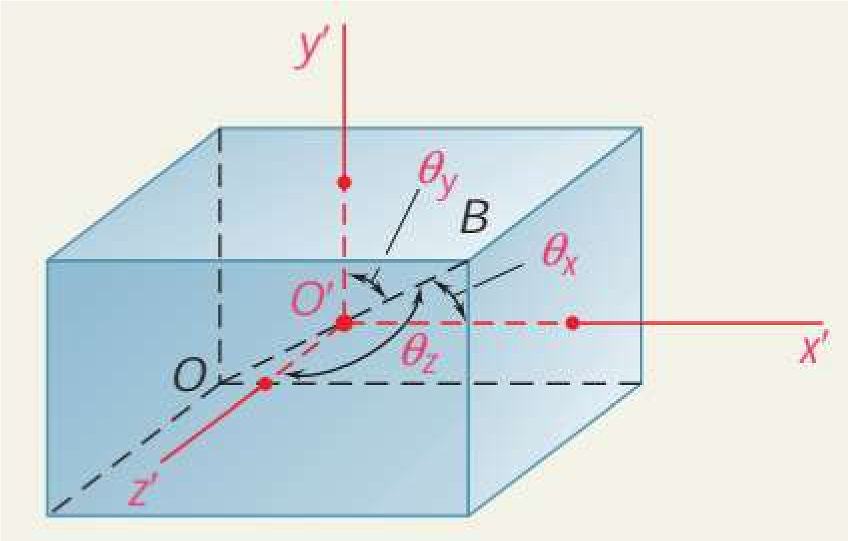
$$\lambda_y = \frac{b}{(a^2 + b^2 + c^2)^{1/2}} \qquad \lambda_z = \frac{c}{(a^2 + b^2 + c^2)^{1/2}}$$

Substituting the values obtained for the moments and products of inertia and for the direction cosines into the equation for I_{OB}, we have

$$I_{OB} = \frac{1}{a^2 + b^2 + c^2}\left[\tfrac{1}{3}m(b^2 + c^2)a^2 + \tfrac{1}{3}m(c^2 + a^2)b^2 + \tfrac{1}{3}m(a^2 + b^2)c^2 - \tfrac{1}{2}ma^2b^2 - \tfrac{1}{2}mb^2c^2 - \tfrac{1}{2}mc^2a^2\right]$$

$$I_{OB} = \frac{m}{6}\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2}$$

Sample Problem 9.14



Sample Problem 9.14

Alternative Solution. The moment of inertia I_{OB} can be obtained directly from the principal moments of inertia $\bar{I}_{x'}, \bar{I}_{y'}, \bar{I}_{z'}$, since the line OB passes through the centroid O'. Since the x', y', z' axes are principal axes of inertia, we use Eq. (9.50) to write

$$
\begin{aligned}
I_{OB} &= \bar{I}_{x'}\lambda_x^2 + \bar{I}_{y'}\lambda_y^2 + \bar{I}_{z'}\lambda_z^2 \\
&= \frac{1}{a^2+b^2+c^2}\left[\frac{m}{12}(b^2+c^2)a^2 + \frac{m}{12}(c^2+a^2)b^2 + \frac{m}{12}(a^2+b^2)c^2\right]
\end{aligned}
$$

$$
I_{OB} = \frac{m}{6}\,\frac{a^2b^2+b^2c^2+c^2a^2}{a^2+b^2+c^2}
$$

Sample Problem 9.15

If $a = 3c$ and $b = 2c$ for the rectangular prism of Sample Prob. 9.14, determine (*a*) the principal moments of inertia at the origin O, (*b*) the principal axes of inertia at O.

Sample Problem 9.15

a. Principal Moments of Inertia at the Origin O. Substituting $a = 3c$ and $b = 2c$ into the solution to Sample Prob. 9.14, we have

$$I_x = \tfrac{5}{3}mc^2 \qquad I_y = \tfrac{10}{3}mc^2 \qquad I_z = \tfrac{13}{3}mc^2$$
$$I_{xy} = \tfrac{3}{2}mc^2 \qquad I_{yz} = \tfrac{1}{2}mc^2 \qquad I_{zx} = \tfrac{3}{4}mc^2$$

Substituting the values of the moments and products of inertia into Eq. (9.56) and collecting terms yields

$$K^3 - (\tfrac{28}{3}mc^2)K^2 + (\tfrac{3479}{144}m^2c^4)K - \tfrac{589}{54}m^3c^6 = 0$$

We then solve for the roots of this equation; from the discussion in Sec. 9.18, it follows that these roots are the principal moments of inertia of the body at the origin.

$$K_1 = 0.568867mc^2 \qquad K_2 = 4.20885mc^2 \qquad K_3 = 4.55562mc^2$$
$$K_1 = 0.569mc^2 \qquad K_2 = 4.21mc^2 \qquad K_3 = 4.56mc^2$$

Sample Problem 9.15

b. Principal Axes of Inertia at *O*. To determine the direction of a principal axis of inertia, we first substitute the corresponding value of K into two of the equations (9.54); the resulting equations together with Eq. (9.57) constitute a system of three equations from which the direction cosines of the corresponding principal axis can be determined. Thus, we have for the first principal moment of inertia K_1:

$$\left(\tfrac{5}{3} - 0.568867\right)mc^2(\lambda_x)_1 - \tfrac{3}{2}mc^2(\lambda_y)_1 - \tfrac{3}{4}mc^2(\lambda_z)_1 = 0$$

$$-\tfrac{3}{2}mc^2(\lambda_x)_1 + \left(\tfrac{10}{3} - 0.568867\right)mc^2(\lambda_y)_1 - \tfrac{1}{2}mc^2(\lambda_z)_1 = 0$$

$$(\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2 = 1$$

Solving yields

$$(\lambda_x)_1 = 0.836600 \qquad (\lambda_y)_1 = 0.496001 \qquad (\lambda_z)_1 = 0.232557$$

Sample Problem 9.15

The angles that the first principal axis of inertia forms with the coordinate axes are then

$$(\theta_x)_1 = 33.2° \qquad (\theta_y)_1 = 60.3° \qquad (\theta_z)_1 = 76.6°$$

Using the same set of equations successively with K_2 and K_3, we find that the angles associated with the second and third principal moments of inertia at the origin are, respectively,

$$(\theta_x)_2 = 57.8° \qquad (\theta_y)_2 = 146.6° \qquad (\theta_z)_2 = 98.0°$$

and

$$(\theta_x)_3 = 82.8° \qquad (\theta_y)_3 = 76.1° \qquad (\theta_z)_3 = 164.3°$$

CHAPTER

10

VECTOR MECHANICS FOR ENGINEERS: STATICS

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Method of Virtual Work

Contents

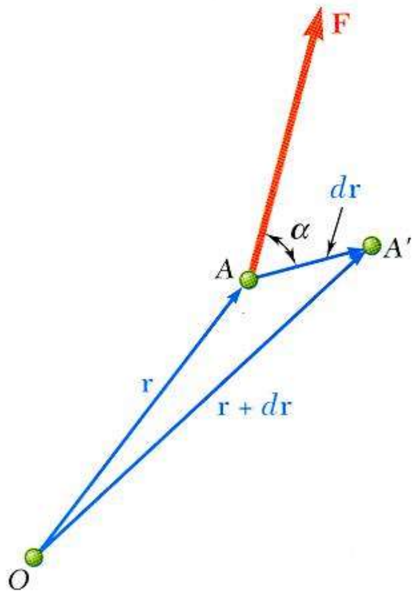
Introduction

- *Principle of virtual work* - if a particle, rigid body, or system of rigid bodies which is in equilibrium under various forces is given an arbitrary *virtual displacement*, the net work done by the external forces during that displacement is zero.

- The principle of virtual work is particularly useful when applied to the solution of problems involving the equilibrium of machines or mechanisms consisting of several connected members.
- If a particle, rigid body, or system of rigid bodies is in equilibrium, then the derivative of its potential energy with respect to a variable defining its position is zero.

- The stability of an equilibrium position can be determined from the second derivative of the potential energy with respect to the position variable.

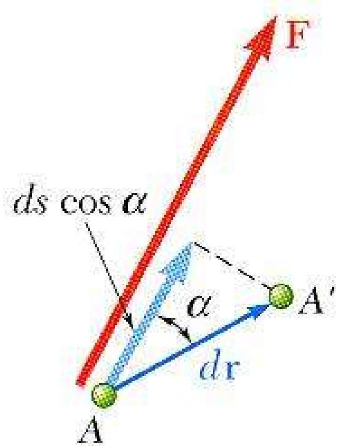
Work of a Force

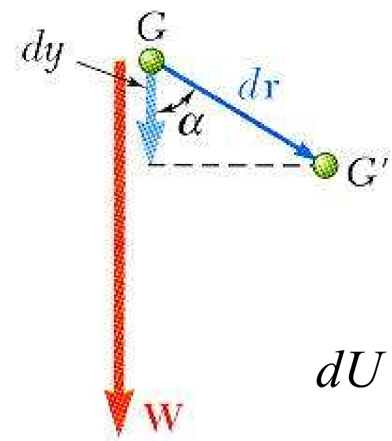


$$d U=\vec{F} \cdot d \vec{r} \quad = \text{ work of the force } \vec{F} \text{ corresponding to the displacement } d\vec{r}$$

$$d U=F\, d s \cos \alpha$$

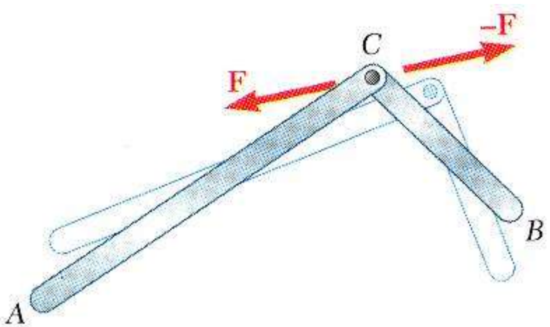
$$\alpha=0,\ d U=+F\, d s \qquad \alpha=\pi,\ d U=-F\, d s \qquad \alpha=\tfrac{\pi}{2},\ d U=0$$

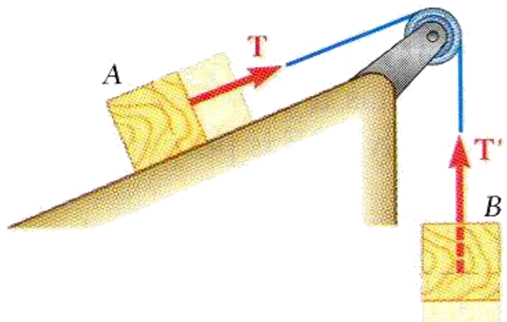




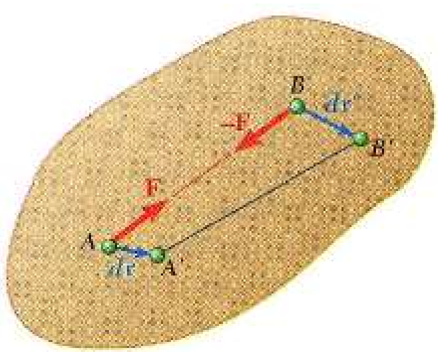
$$d U=W d y$$

Work of a Force





(b)



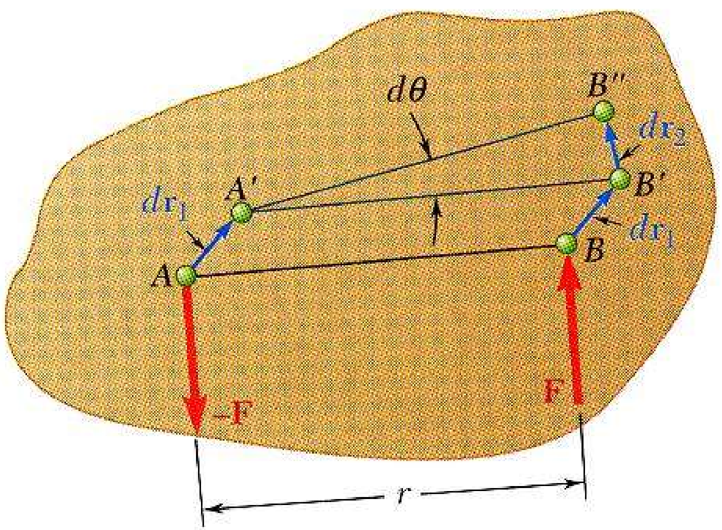
Forces which do no work:

- reaction at a frictionless pin due to rotation of a body around the pin
- reaction at a frictionless surface due to motion of a body along the surface
- weight of a body with cg moving horizontally
- friction force on a wheel moving without slipping

Sum of work done by several forces may be zero:

- bodies connected by a frictionless pin
- bodies connected by an inextensible cord
- internal forces holding together parts of a rigid body

Work of a Couple

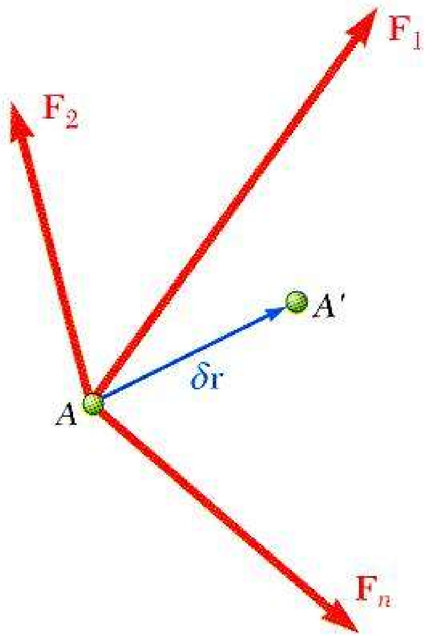


Small displacement of a rigid body:

- translation to *A'B'*
- rotation of *B'* about *A'* to *B''*

$$
\begin{aligned}
W &= -\vec{F} \cdot d\vec{r}_1 + \vec{F} \cdot \left(d\vec{r}_1 + d\vec{r}_2\right) \\
&= \vec{F} \cdot d\vec{r}_2 = F\, ds_2 = F\, r d\theta \\
&= M\, d\theta
\end{aligned}
$$

Principle of Virtual Work



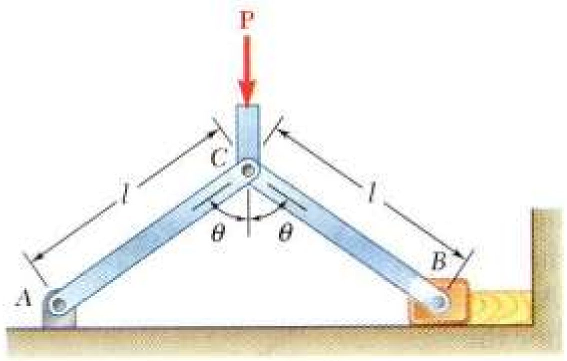
- *Imagine* the small *virtual displacement* of particle which is acted upon by several forces.
- The corresponding *virtual work*,

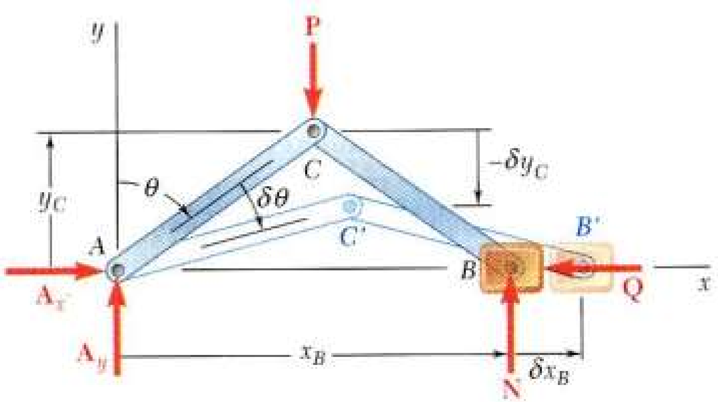
$$\delta U = \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + \vec{F}_3 \cdot \delta\vec{r} = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot \delta\vec{r}$$
$$= \vec{R} \cdot \delta\vec{r}$$

Principle of Virtual Work:

- If a particle is in equilibrium, the total virtual work of forces acting on the particle is zero for any virtual displacement.
- If a rigid body is in equilibrium, the total virtual work of external forces acting on the body is zero for any virtual displacement of the body.
- If a system of connected rigid bodies remains connected during the virtual displacement, only the work of the external forces need be considered.

Applications of the Principle of Virtual Work





- Wish to determine the force of the vice on the block for a given force *P*.
- Consider the work done by the external forces for a virtual displacement $\delta\theta$. Only the forces *P* and *Q* produce nonzero work.

$$\delta U = 0 = \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C$$

$$x_B = 2l\sin\theta \qquad y_C = l\cos\theta$$

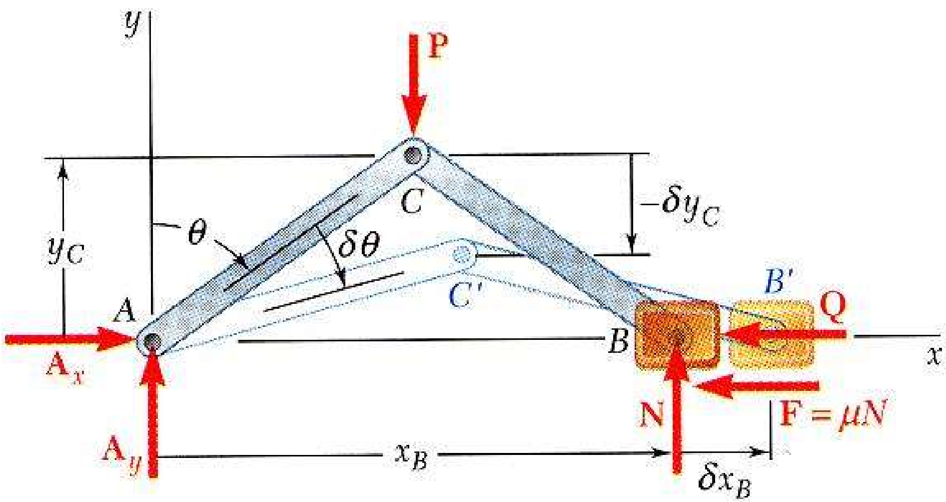
$$\delta x_B = 2l\cos\theta\,\delta\theta \qquad \delta y_C = -l\sin\theta\,\delta\theta$$

$$0 = -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta$$

$$Q = \tfrac{1}{2}P\tan\theta$$

- If the virtual displacement is consistent with the constraints imposed by supports and connections, only the work of loads, applied forces, and friction forces need be considered.

Real Machines. Mechanical Efficiency



- For an ideal machine without friction, the output work is equal to the input work.

- When the effect of friction is considered, the output work is reduced.

$$\delta U = -Q\delta x_B - P\delta y_C - F\delta x_B = 0$$

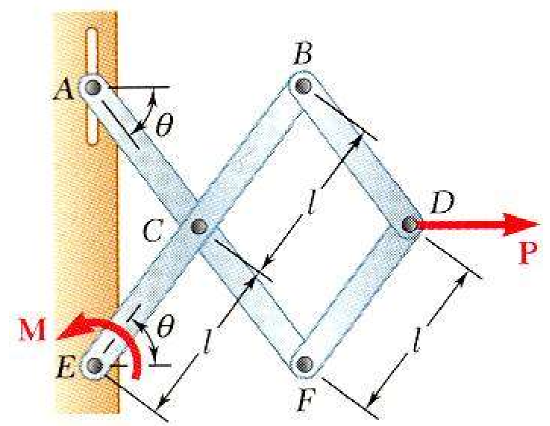
$$0 = -2Ql\cos\theta\delta\theta + Pl\sin\theta\delta\theta - \mu Pl\cos\theta\delta\theta$$

$$Q = \tfrac{1}{2}P(\tan\theta - \mu)$$

$$\eta = \text{mechanical efficiency} = \frac{\text{output work of actual machine}}{\text{output work of ideal machine}}$$

$$\eta = \frac{\text{output work}}{\text{input work}} = \frac{2Ql\cos\theta\delta\theta}{Pl\sin\theta\delta\theta} = 1 - \mu\cot\theta$$

Sample Problem 10.1



Determine the magnitude of the couple M required to maintain the equilibrium of the mechanism.

SOLUTION:

- Apply the principle of virtual work

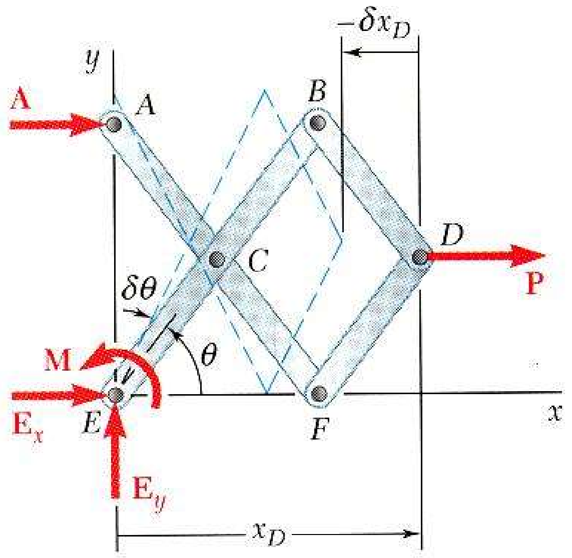
$$\delta U = 0 = \delta U_M + \delta U_P$$

$$0 = M\delta\theta + P\delta x_D$$

$$x_D = 3l\cos\theta$$

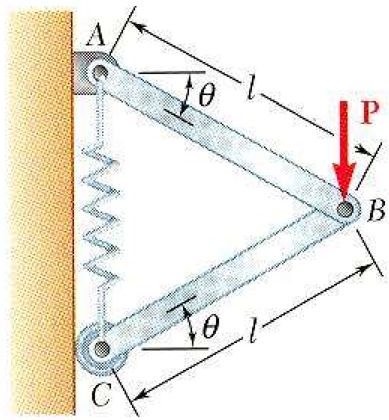
$$\delta x_D = -3l\sin\theta\delta\theta$$

$$0 = M\delta\theta + P(-3l\sin\theta\delta\theta)$$

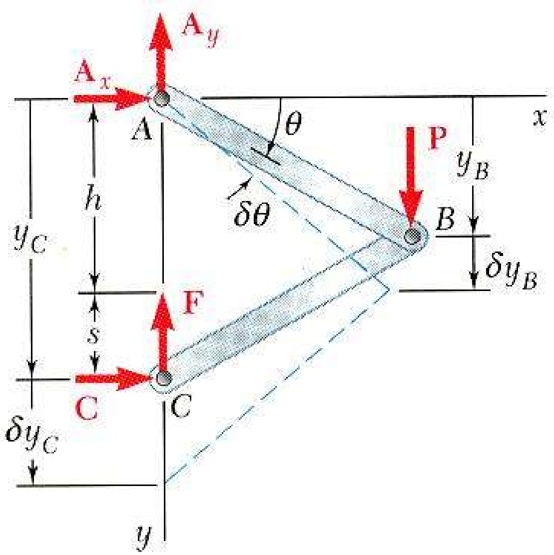


$$\boxed{M = 3Pl\sin\theta}$$

Sample Problem 10.2



Determine the expressions for θ and for the tension in the spring which correspond to the equilibrium position of the spring. The unstretched length of the spring is h and the constant of the spring is k. Neglect the weight of the mechanism.



SOLUTION:

- Apply the principle of virtual work

$$\delta U = \delta U_B + \delta U_F = 0$$

$$0 = P\delta y_B - F\delta y_C$$

$$y_B = l\sin\theta \qquad y_C = 2l\sin\theta \qquad F = ks$$

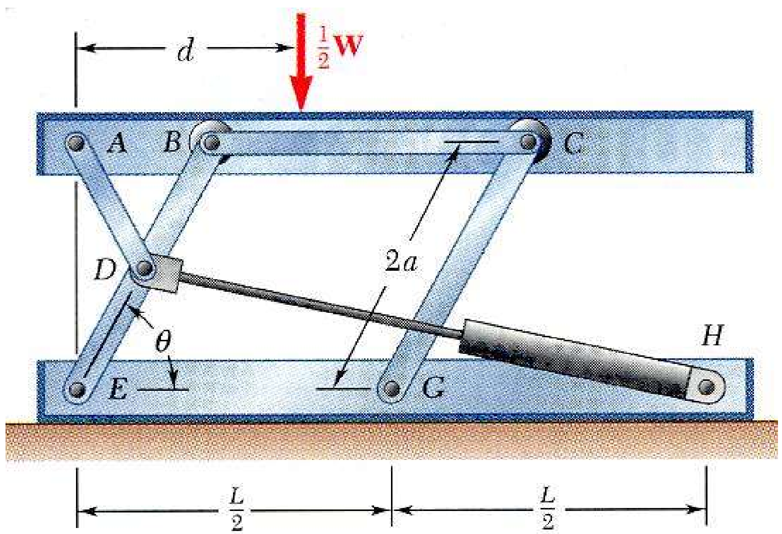
$$\delta y_B = l\cos\theta\delta\theta \qquad \delta y_C = 2l\cos\theta\delta\theta \qquad = k(y_C - h)$$

$$= k(2l\sin\theta - h)$$

$$0 = P(l\cos\theta\delta\theta) - k(2l\sin\theta - h)(2l\cos\theta\delta\theta)$$

$$\boxed{\sin\theta = \frac{P + 2kh}{4kl} \qquad F = \tfrac{1}{2}P}$$

Sample Problem 10.3

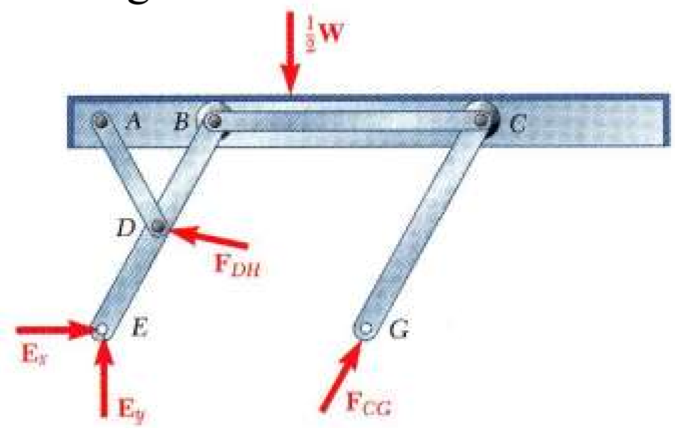


A hydraulic lift table consisting of two identical linkages and hydraulic cylinders is used to raise a 1000-kg crate. Members *EDB* and *CG* are each of length 2*a* and member *AD* is pinned to the midpoint of *EDB*.

Determine the force exerted by each cylinder in raising the crate for $\theta = 60^o$, $a = 0.70$ m, and $L = 3.20$ m.

SOLUTION:

- Create a free-body diagram for the platform and linkage.

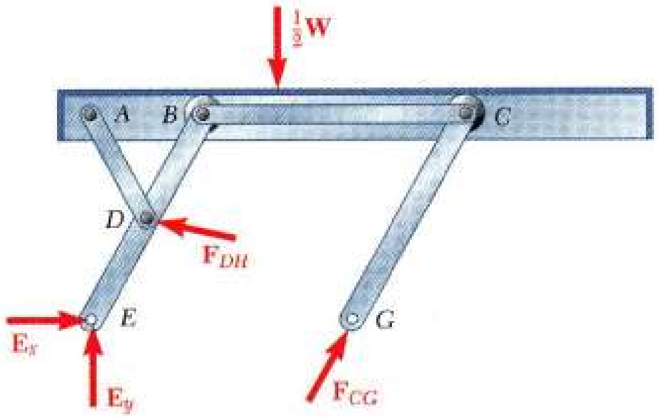


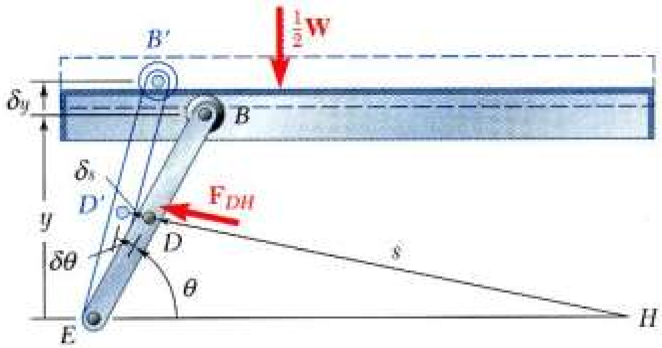
- Apply the principle of virtual work for a virtual displacement $\delta\theta$ recognizing that only the weight and hydraulic cylinder do work.

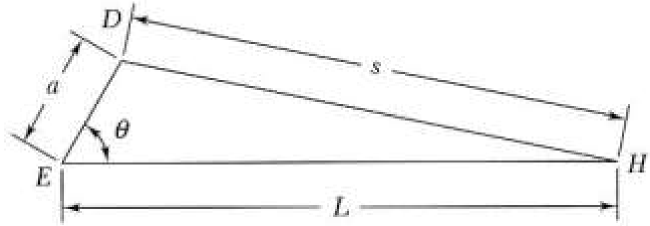
$$\delta U = 0 = \delta Q_W + \delta Q_{F_{DH}}$$

- Based on the geometry, substitute expressions for the virtual displacements and solve for the force in the hydraulic cylinder.

Sample Problem 10.3







SOLUTION:

- Create a free-body diagram for the platform.
- Apply the principle of virtual work for a virtual displacement $\delta\theta$

$$\delta U = 0 = \delta Q_W + \delta Q_{F_{DH}}$$

$$0 = -\tfrac{1}{2} W \delta y + F_{DH} \delta s$$

- Based on the geometry, substitute expressions for the virtual displacements and solve for the force in the hydraulic cylinder.

$$y = 2a\sin\theta$$

$$\delta y = 2a\cos\theta\delta\theta$$

$$s^2 = a^2 + L^2 - 2aL\cos\theta$$

$$2s\delta s = -2aL(-\sin\theta)\delta\theta$$

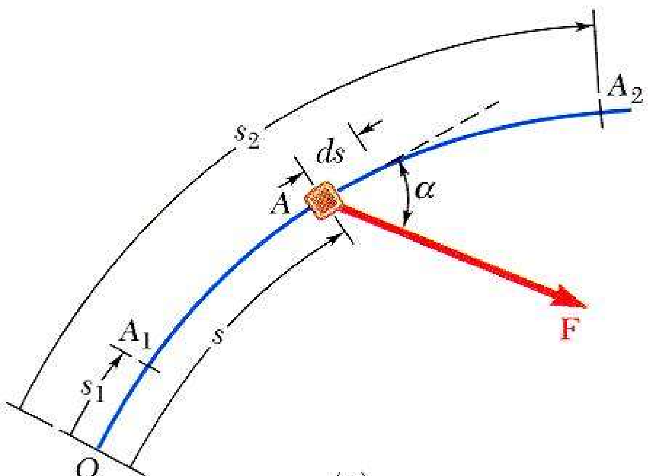
$$\delta s = \frac{aL\sin\theta}{s}\delta\theta$$

$$0 = \left(-\tfrac{1}{2}W\right)2a\cos\theta\delta\theta + F_{DH}\frac{aL\sin\theta}{s}\delta\theta$$

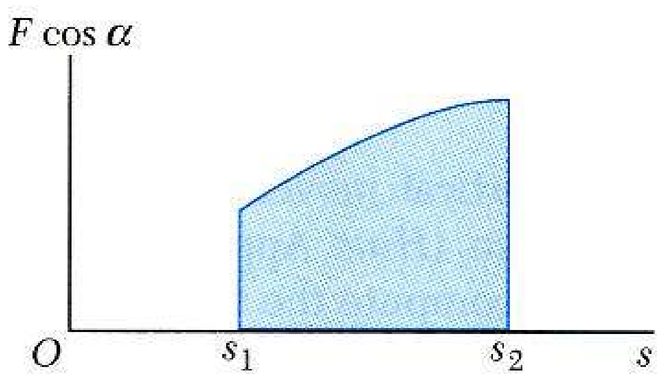
$$F_{DH} = W\frac{s}{L}\cot\theta$$

$$\boxed{F_{DH} = 5.15\,\text{kN}}$$

Work of a Force During a Finite Displacement



(a)



(b)

- Work of a force corresponding to an infinitesimal displacement,

$$dU = \vec{F} \cdot d\vec{r}$$
$$= F\, ds \cos\alpha$$

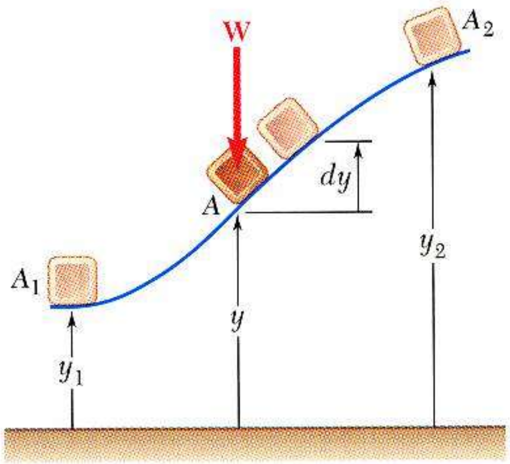
- Work of a force corresponding to a finite displacement,

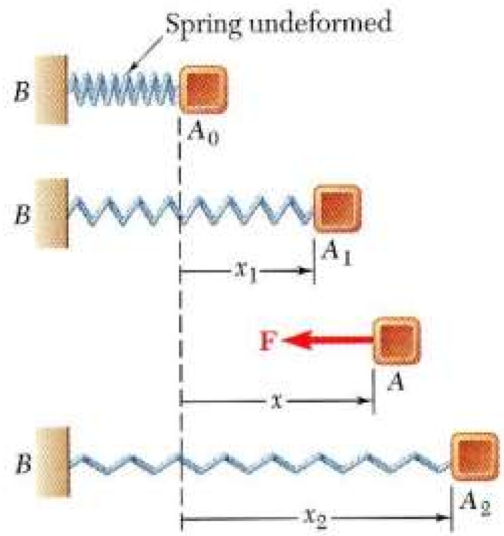
$$U_{1\to 2} = \int_{s_1}^{s_2} (F\cos\alpha)\, ds$$

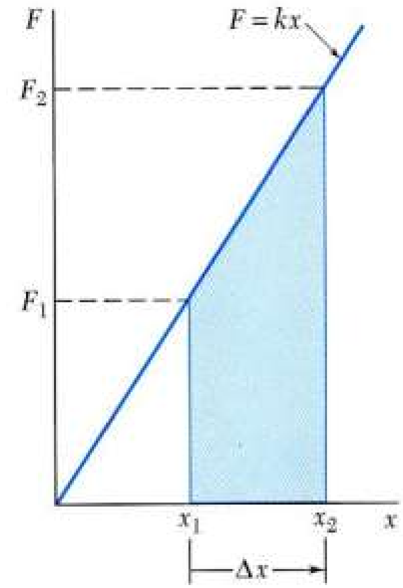
- Similarly, for the work of a couple,

$$dU = M d\theta$$
$$U_{1\to 2} = M(\theta_2 - \theta_1)$$

Work of a Force During a Finite Displacement







Work of a weight,

$$dU = -W dy$$

$$U_{1\to 2} = -\int_{y_1}^{y_2} W dy$$

$$= W y_1 - W y_2$$

$$= -W \Delta y$$

Work of a spring,

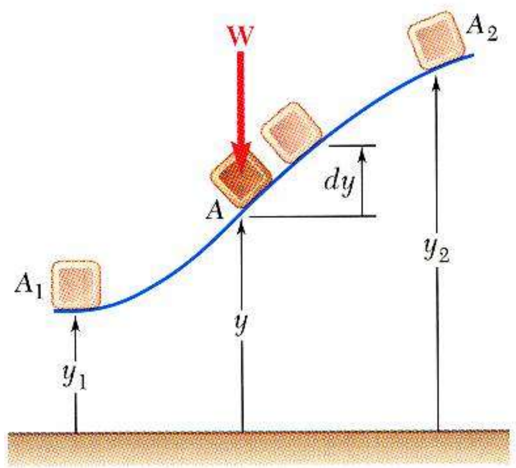
$$dU = -F dx = -(kx) dx$$

$$U_{1\to 2} = -\int_{x_1}^{x_2} kx\, dx$$

$$= \tfrac{1}{2} k x_1^2 - \tfrac{1}{2} k x_2^2$$

$$U_{1\to 2} = -\tfrac{1}{2}(F_1 + F_2)\Delta x$$

Potential Energy



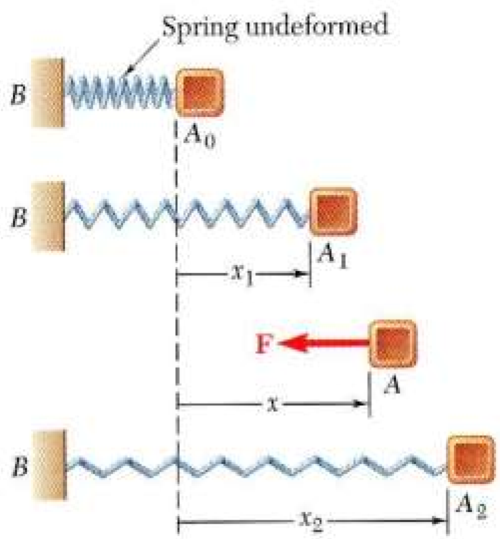
- Work of a weight

$$U_{1\to 2} = Wy_1 - Wy_2$$

The work is independent of path and depends only on

$$Wy = V_g = \textit{potential energy} \text{ of the body with respect to the } \textit{force of gravity } \vec{W}$$

$$U_{1\to 2} = \left(V_g\right)_1 - \left(V_g\right)_2$$



- Work of a spring,

$$U_{1\to 2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$
$$= \left(V_e\right)_1 - \left(V_e\right)_2$$

$$V_e = \textit{potential energy} \text{ of the body with respect to the } \textit{elastic force } \vec{F}$$

Potential Energy

- When the differential work is a force is given by an exact differential,

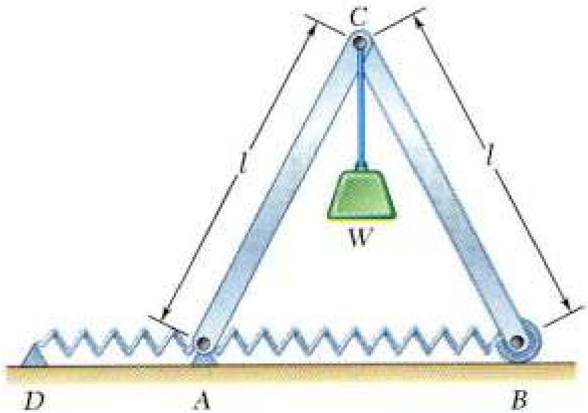
$$dU = -dV$$

$$U_{1\to 2} = V_1 - V_2$$

$$= negative\ of\ change\ in\ potential\ energy$$

- Forces for which the work can be calculated from a change in potential energy are *conservative forces*.

Potential Energy and Equilibrium



- When the potential energy of a system is known, the principle of virtual work becomes

$$\delta U = 0 = -\delta V = -\frac{dV}{d\theta}\delta\theta$$

$$0 = \frac{dV}{d\theta}$$

- For the structure shown,

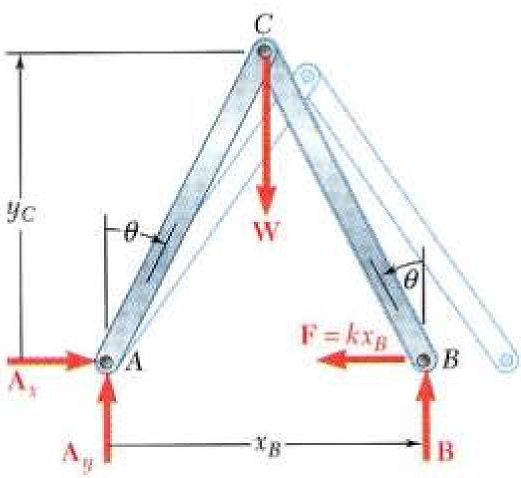
$$V = V_e + V_g = \tfrac{1}{2}kx_B^2 + Wy_C$$

$$= \tfrac{1}{2}k(2l\sin\theta)^2 + W(l\cos\theta)$$

- At the position of equilibrium,

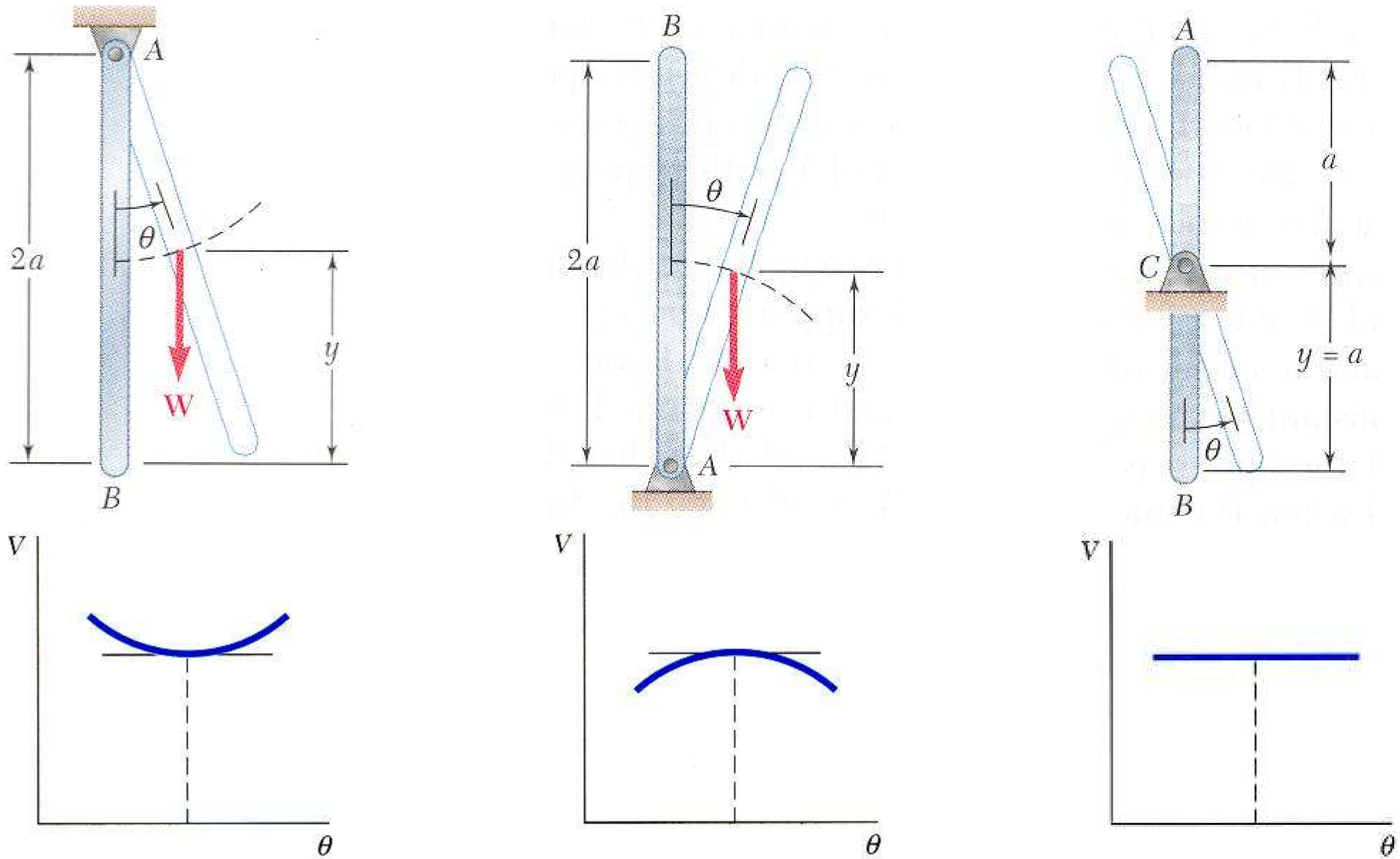
$$\frac{dV}{d\theta} = 0 = l\sin\theta(4kl\cos\theta - W)$$

indicating two positions of equilibrium.



Stability of Equilibrium

$$\frac{dV}{d\theta} = 0$$



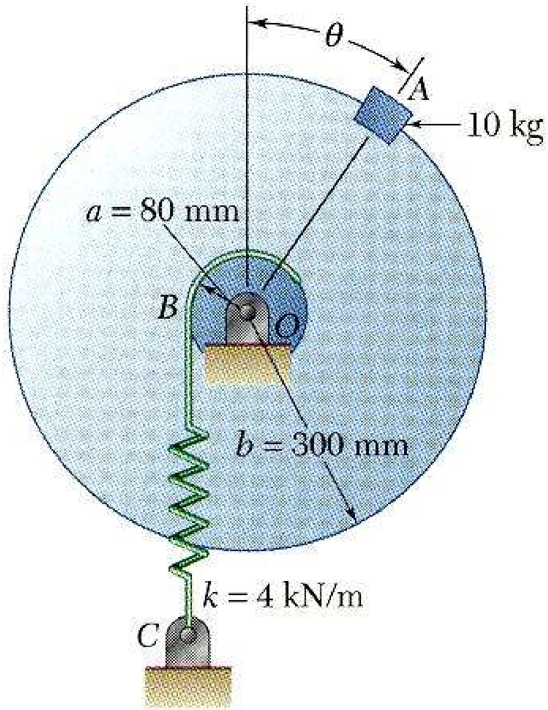
(a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

$$\frac{d^2V}{d\theta^2} > 0$$

$$\frac{d^2V}{d\theta^2} > 0$$

Must examine higher order derivatives.

Sample Problem 10.4



Knowing that the spring *BC* is unstretched when $\theta = 0$, determine the position or positions of equilibrium and state whether the equilibrium is stable, unstable, or neutral.

SOLUTION:

- Derive an expression for the total potential energy of the system.

$$V = V_e + V_g$$

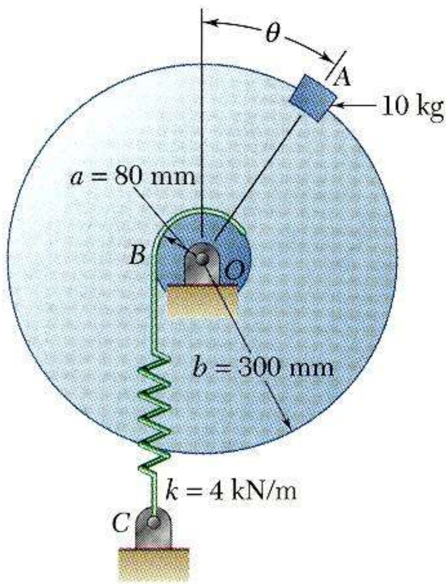
- Determine the positions of equilibrium by setting the derivative of the potential energy to zero.

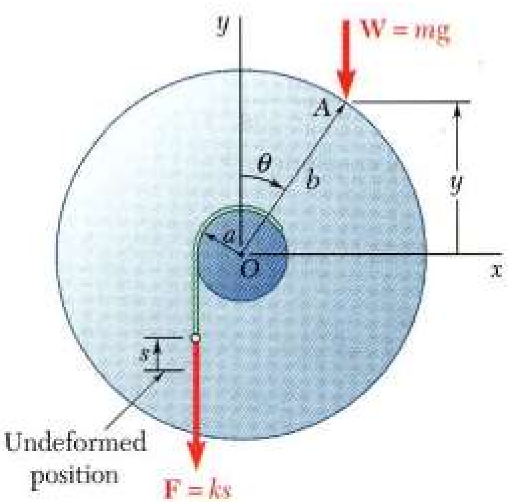
$$\frac{dV}{d\theta} = 0$$

- Evaluate the stability of the equilibrium positions by determining the sign of the second derivative of the potential energy.

$$\frac{d^2V}{d\theta^2} \overset{?}{><} 0$$

Sample Problem 10.4





SOLUTION:

- Derive an expression for the total potential energy of the system.

$$V = V_e + V_g$$
$$= \frac{1}{2}ks^2 + mgy$$
$$= \frac{1}{2}k(a\theta)^2 + mg(b\cos\theta)$$

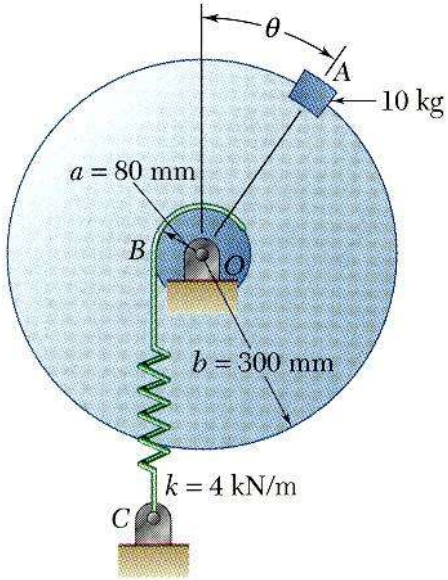
- Determine the positions of equilibrium by setting the derivative of the potential energy to zero.

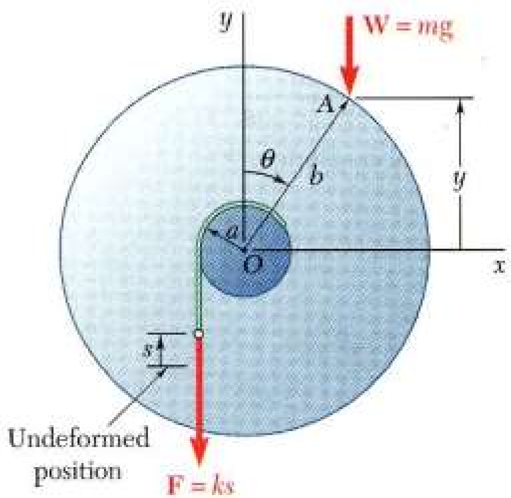
$$\frac{dV}{d\theta} = 0 = ka^2\theta - mgb\sin\theta$$

$$\sin\theta = \frac{ka^2}{mgb}\theta = \frac{(4\,\text{kN/m})(0.08\text{m})^2}{(10\text{kg})(9.81\text{m/s}^2)(0.3\text{m})}\theta$$
$$= 0.8699\theta$$

$$\boxed{\theta = 0 \qquad \theta = 0.902\,\text{rad} = 51.7°}$$

Sample Problem 10.4





- Evaluate the stability of the equilibrium positions by determining the sign of the second derivative of the potential energy.

$$V = \tfrac{1}{2}k(a\theta)^2 + mg(b\cos\theta)$$

$$\frac{dV}{d\theta} = 0 = ka^2\theta - mgb\sin\theta \qquad \theta = 0$$

$$\theta = 0.902\,\text{rad} = 51.7°$$

$$\frac{d^2V}{d\theta^2} = ka^2 - mgb\cos\theta$$

$$= (4\,\text{kN/m})(0.08\text{m})^2 - (10\text{kg})\left(9.81\,\text{m/s}^2\right)(0.3\text{m})\cos\theta$$

$$= 25.6 - 29.43\cos\theta$$

at $\theta = 0$: $\dfrac{d^2V}{d\theta^2} = -3.83 < 0$ $\boxed{\text{unstable}}$

at $\theta = 51.7°$: $\dfrac{d^2V}{d\theta^2} = +7.36 > 0$ $\boxed{\text{stable}}$